18.726 Problem Set 1, due Thursday, February 10

Please submit *ten* of the following problems, including all problems marked "Required" and exactly two of the problems from Hartshorne II.1. You are encouraged to work together, but you must write up your own solutions; better yet, try not to write up the same set of ten problems as anyone you worked with!

Also, please provide an estimate of how much time you spent on this; if this set ends up being too long, I'll try to tone things down in the future.

1. Hartshorne II.1.3.

2. Hartshorne II.1.13.

3. Hartshorne II.1.15.

4. Hartshorne II.1.18 (look at this one if you're queasy about the term "adjoint functors").

5. Hartshorne II.1.19.

6. (Required) Describe the *closed* points of $\operatorname{Spec} K[x, y]$ and of $\operatorname{Spec} K[x, y, z]$ in each of the following cases:
   (a) $K = \mathbb{C}$;
   (b) $K = \mathbb{R}$;
   (c) $K = \mathbb{F}_p$; (Hint: remember that $\mathbb{F}_p$ has exactly one extension of each degree)
   (d) (optional) $K = \mathbb{Q}$.

7. Hartshorne II.2.3.

8. (a) Let $(X, \mathcal{O}_X)$ be a locally ringed space. Prove that for any $f \in \Gamma(X, \mathcal{O}_X)$, the set of $x \in X$ such that $f \notin \mathfrak{m}_x$ is open. (Here $\mathfrak{m}_x$ denotes the maximal ideal of the local ring $\mathcal{O}_x = \mathcal{O}_{X,x}$, i.e., the stalk of the sheaf $\mathcal{O}_X$ at x.)
   (b) Let A be a ring, and let $(X, \mathcal{O}_X)$ be a locally ringed space. Prove that the "global sections" map
   $$\operatorname{Hom}_{\mathrm{LocRingSp}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X))$$
   is a bijection. In other words, the "affine scheme" functor from (the opposite category of) rings to locally ringed spaces is not so crazy at all: it's adjoint to the global sections functor! (Compare Proposition II.2.3, which is the case where $(X, \mathcal{O}_X)$ is an affine scheme, and problem II.2.4, which is the case where $(X, \mathcal{O}_X)$ is a general scheme.)

9. Hartshorne II.2.5.

10. (Required) Hartshorne II.2.8.

11. (Required) Hartshorne II.2.13.
12. (Required) Hartshorne II.2.14.
13. Hartshorne II.2.16.
14. Hartshorne II.3.6. (See problem II.2.9 for the definition of "generic point".)
15. Hartshorne II.3.16.
16. (don't actually submit this one, but do think about it) Hartshorne alludes briefly to the point of view that a presheaf (of sets) on a topological space X is a contravariant functor from the category $\mathrm{Top}(X)$ to the category of sets. That observation triggered one of Grothendieck's insights: you don't need the full strength of topology in order to talk about sheaves.

    (a) For starters, try to define the notion of a "sheaf" on the category of *all* topological spaces. (Hint: there should be at least one nontrivial example, namely the sheaf of continuous functions!)

    (b) Given a topological space X, let $\mathrm{Cov}(X)$ be the category whose objects are covering space maps $V \to U$, with $U \subseteq X$ open. Define a "sheaf" on that category, then give a simpler description of what you actually just constructed. (This example is a model for the étale topology on a scheme.)

    (c) Try to axiomatize the conditions on a category under which sheaves are defined. More precisely, given a category $\mathcal{C}$ in which finite fibred products exist, for $X \in \mathrm{Obj}(\mathcal{C})$, a "covering of X" should be a collection of arrows $\{f_i : U_i \to X\}_{i \in I}$; however, you should further axiomatize what it means for a covering to be "admissible", and then the sheaf axiom should only apply to admissible coverings. (Hint: Grothendieck's three rules are: the trivial covering $\{U \to U\}$ should be admissible; the pullback of a trivial covering should be admissible; it should be possible to "compose" admissible coverings.) I may discuss this construction later in the course, if time permits.