Not 18.726 Problem Set 8; not to be turned in

In case you aren't familiar with homological algebra, you should try some of these.

1. Verify the *five lemma*: in an abelian category (which you may assume admits a faithful functor to abelian groups), if the diagram

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

   has exact rows, then:

   (a) If f_1 is an epimorphism and f_2, f_4 are monomorphisms, then f_3 is a monomorphism.

   (b) If f_5 is a monomorphism and f_2, f_4 are epimorphisms, then f_3 is an epimorphism.

2. Suppose the abelian category $\mathcal{C}$ has enough injectives. Prove that any morphism $X \to Y$ in $\mathcal{C}$ extends to a morphism between any injective resolutions of X and Y, and that any two such extensions are homotopic. (I more or less did this in class, but make sure you understood!)

3. Verify Theorem 1.1A; this amounts to checking that the long exact sequence associated to a short exact sequence of complexes really is an exact sequence.

4. Verify Proposition 1.2A.

5. Teach yourself what a "spectral sequence" is, or at least what the Leray spectral sequence associated to a double complex is. (A friendly first reference point might be Bott and Tu, "Differential Forms in Algebraic Topology", but don't put too much faith in correctness of the formulas there.) Then figure out why that construction gives a quick proof of Proposition 1.2A.

6. Verify Theorem 1.3A; feel free to assume the source category has a faithful functor (so that diagram-chasing is legal).