18.726 Problem Set 9, due April 14

Do any *six* problems, including the ones marked "Required". (TeX note: the haček-topped letter Ȟ is produced by `\v{H}` in text mode and `\check{H}` in math mode.)

1. Hartshorne III.2.1, part (a) (part (b) optional).
2. Hartshorne III.2.2.
3. Hartshorne III.3.1.
4. Hartshorne III.3.6.
5. Hartshorne III.4.2.
6. Hartshorne III.4.3.
7. Hartshorne III.4.9 (nice geometric application of cohomology!).
8. Determine $\dim_k H^i(\mathbb{P}^n, \mathcal{O}(m))$ for all i, m in case $n = 1$ (I sketched this in class) and $n = 2$ (similar idea, but also see the proof of Theorem III.5.1).
9. Calculate directly the Čech cohomology of the structure sheaf $\mathcal{O}$ on $\operatorname{Spec} k[x]$, relative to the cover by the two affines $\operatorname{Spec} k[x, 1/x]$ and $\operatorname{Spec} k[x, 1/(x-1)]$; this should give you some idea of how Grothendieck's general argument works.
10. (Required unless you are completely stuck) Let $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots$ be a complex of sheaves on a topological space X. Given a covering $\{U_i\}$ of X, the *Čech hypercohomology* of the complex is the cohomology of the double complex

$$D^{ij} = C^i(\mathcal{F}_j)$$

where C^i is the usual Čech cohomology group. (To take cohomology of a double complex, "flatten" it into a single complex: make the complex E with $E^k = \oplus_{i+j} D^{ij}$ with differential $d_i + (-1)^i d_j$, where d_i and d_j are the differentials in the i and j directions.) Compute the hypercohomology of the de Rham complex (i.e., $\Omega^i = \wedge^i \Omega^1$, the exterior product taken over $\mathcal{O}_X$) in the following cases, using any open affine cover you want; the result is called the *algebraic de Rham cohomology* of X. (Optional: justify why the answers don't depend on the choice of the cover!)

   (a) $X = \mathbb{P}^2_k$, for k any field.

   (b) X is the closure in $\mathbb{P}^2_k$ of the affine hyperelliptic curve $y^2 = P(x)$, where P is a polynomial of degree $2g + 1$ with no repeated roots, and k has characteristic not 2. (This time you really will be able to get by without assuming characteristic zero!)

11. Hartshorne III.4.4. (Not required, but it will help you on the next one; also, it includes some facts I stated in class about Čech cohomology and refinements.)

12. (Required, but not as bad as it looks!) Let X be a quasi-compact topological space, let B be a basis of opens closed under finite intersections (including the empty intersection, so $X \in B$), and let $\mathcal{F}$ be a sheaf of abelian groups on X. (Case of interest: X an affine scheme, B distinguished opens, and $\mathcal{F}$ quasicoherent.)

    (a) Let $D^p(\mathcal{F})$ denote the direct limit of the Čech chain groups C^p (i.e., the product of the sections over each $(p+1)$-fold intersection) over all finite coverings by elements of B. Prove that if you start with a short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, the resulting sequence $0 \to D^{\cdot}(\mathcal{F}) \to D^{\cdot}(\mathcal{G}) \to D^{\cdot}(\mathcal{H}) \to 0$ is again exact. (Note that the surjectivity at the right fails if you try to do this on a fixed covering: see Caution III.4.0.2.) Deduce as corollary that $\check{H}^i$ form a δ-functor. (Note that this gives a map $H^i \to \check{H}^i$, but that's not enough to show that these are isomorphic.)

    (b) Let $\check{H}^i(X, \mathcal{F})$ be the the direct limit of Čech cohomology over all coverings of X by elements of B. (See III.4.4(a) for the precise definition of this direct system.) Prove that the natural map $\check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F})$ is always an isomorphism. (Hint: see III.4.4(c). Better yet, stick $\mathcal{F}$ into an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ with $\mathcal{G}$ flasque, and hence acyclic, then consider the diagram

$$\begin{array}{ccccccccc} \cdots \to & \check{H}^i(X,\mathcal{F}) & \to & \check{H}^i(X,\mathcal{G}) & \to & \check{H}^i(X,\mathcal{H}) & \to & \check{H}^{i+1}(X,\mathcal{F}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H^i(X,\mathcal{F}) & \to & H^i(X,\mathcal{G}) & \to & H^i(X,\mathcal{H}) & \to & H^{i+1}(X,\mathcal{F}) & \to \cdots \end{array}$$

    and remember the five lemma.)

    (c) Suppose that $\check{H}^i(U, \mathcal{F}) = 0$ for all $U \in B$ and all $i > 0$. Prove that for all $i > 0$, $H^i(X, \mathcal{F}) = 0$. (Hint: in notation of the previous hint, verify that $\mathcal{H}$ also satisfies the input condition. Also, remember that anything you can prove about X is also true about any $U \in B$.)