

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Differentials

See Hartshorne II.8.

1 The module of Kähler differentials

Let $A \rightarrow B$ be a homomorphism of rings. The *module of Kähler differentials* of B over A is a B -module $\Omega_{B/A}$ equipped with an A -linear derivation $d : B \rightarrow \Omega_{B/A}$ (an A -linear homomorphism satisfying the Leibniz rule $d(xy) = x dy + y dx$ for $x, y \in B$; note that this forces $d(a) = 0$ for $a \in A$), with the following universal property: for any B -module M and any A -linear derivation $\partial : B \rightarrow M$, ∂ factors uniquely through d via a B -linear homomorphism $\Omega_{B/A} \rightarrow M$.

There are two standard ways to construct $\Omega_{B/A}$. One is to form the B -module generated by symbols db for $b \in B$, modulo the necessary relations:

- (a) $d(b_1 b_2) - b_1 db_2 - b_2 db_1$ for $b_1, b_2 \in B$;
- (b) $d(b_1 + b_2) = d(b_1) + d(b_2)$ for $b_1, b_2 \in B$;
- (c) $d(a) = 0$ for $a \in A$.

This obviously has the desired universal property. The other is to let I be the kernel of the multiplication map $B \otimes_A B \rightarrow B$, and put $\Omega_{B/A} = I/I^2$ equipped with the map $d(b) = b \otimes 1 - 1 \otimes b$. This evidently gives an A -linear derivation. Given a derivation $\partial : B \rightarrow M$, view $B \oplus M$ as a B -algebra in by setting $m_1 m_2 = 0$ for all $m_1, m_2 \in M$. Then the formula

$$b_1 \otimes b_2 \rightarrow (b_1 b_2, b_1 \partial(b_2))$$

induces a ring homomorphism $B \otimes_A B \rightarrow B \oplus M$ under which I maps to M , so I^2 maps to 0 and we get a B -linear map $I/I^2 \rightarrow M$. Composing with d easily gives back ∂ . The uniqueness of the factorization follows by observing that

$$x \otimes y = xy \otimes 1 - x(y \otimes 1 - 1 \otimes y)$$

so the image of d generates I (and hence I/I^2) as a B -module.

For instance, if $B = A[x_1, \dots, x_n]$, then $\Omega_{B/A}$ is freely generated by dx_1, \dots, dx_n . Also, if k is an algebraically closed field and A is a reduced quotient of $k[x_1, \dots, x_n]$, then the Jacobian criterion can be interpreted as saying that A corresponds to a nonsingular variety over k if and only if $\Omega_{A/k}$ is locally free as an A -module.

For another example, if A is a field and B is a finite field extension, then $\Omega_{B/A} = 0$ if and only if B is *separable* over A .

2 The sheaf of Kähler differentials

Let $f : Y \rightarrow X$ be a morphism. For each open affine subset $U = \operatorname{Spec}(A)$ of X and each open affine subset $V = \operatorname{Spec}(B)$ of $f^{-1}(U)$, form the module $\Omega_{B/A}$. We would like these to form the sections of a sheaf $\Omega_{Y/X}$, but checking the glueing property directly from this definition is a bit awkward.

Fortunately, our second construction of the module of Kähler differentials suggests a global definition of the sheaf $\Omega_{Y/X}$. We'll explain this first in case f is separated. In that case, $\Delta : Y \rightarrow Y \times_X Y$ is a closed immersion; let \mathcal{I} be the corresponding ideal sheaf on $Y \times_X Y$. We then put

$$\Omega_{Y/X} = \Delta^*(\mathcal{I}/\mathcal{I}^2).$$

But what if f is not separated? In that case, we still claim that Δ is an *immersion*; this follows from the proof of Hartshorne Corollary II.4.2. Then Δ gives rise to an ideal sheaf not on $Y \times_X Y$, but on some open subscheme containing the image of Δ ; we may then proceed as in the separated case.

Useful properties of Ω :

- The formation of $\Omega_{Y/X}$ commutes with base change as follows. If $g : Z \rightarrow X$ is another morphism, then $\Omega_{Y \times_X Z/Z}$ can be identified canonically with the pullback of $\Omega_{Y/X}$ along the projection $Y \times_X Z \rightarrow Y$ (Hartshorne, Proposition II.8.10).
- If $f : Z \rightarrow Y$ and $g : Y \rightarrow X$ are morphisms, then there is a natural exact sequence

$$f^*\Omega_{Y/X} \rightarrow \Omega_{Z/X} \rightarrow \Omega_{Z/Y} \rightarrow 0$$

(Hartshorne, Proposition II.8.11).

- If $f : Y \rightarrow X$ is a morphism, and $j : Z \rightarrow Y$ is the closed immersion defined by the ideal sheaf \mathcal{I} on Y , then there is a natural exact sequence of sheaves on Z :

$$j^*(\mathcal{I}/\mathcal{I}^2) \rightarrow j^*(\Omega_{Y/X}) \rightarrow \Omega_{Z/X} \rightarrow 0$$

(Hartshorne, Proposition II.8.12).

- Let A be a ring, and let $f : Y = \mathbb{P}_A^n \rightarrow X = \operatorname{Spec} A$ be the natural morphism. We then have a short exact sequence

$$0 \rightarrow \Omega_{Y/X} \rightarrow \mathcal{O}_X(-1)^{\oplus(n+1)} \rightarrow \mathcal{O}_Y \rightarrow 0$$

(Hartshorne, Theorem II.8.13).

As in the affine case, a variety X over a field k is nonsingular if and only if $\Omega_{X/k}$ is locally free. Since $\Omega_{X/k}$ is necessarily finitely generated (deduce this from the case of affine space), there is always an open dense subset U of X which is nonsingular over k .

Suppose X is nonsingular of dimension n (on each component). Then we call the sheaf $\omega_{X/k} = \wedge^n \Omega_{X/k}$ the *canonical sheaf* on X ; it is locally free of rank 1. As the name suggests,

the canonical sheaf is an omnipresent object in the study of the geometry of varieties; we will see it in the Riemann-Roch theorem, and more generally in Serre duality, but it is also a central player in modern birational geometry, as in the following *very hard* theorem.

Theorem (Bircar-Cascini-Hacon-McKernan, Siu). *Let X be a smooth projective irreducible variety over \mathbb{C} . Then the ring*

$$\bigoplus_{n=0}^{\infty} \Gamma(X, \omega_{X/k}^{\otimes n})$$

is finitely generated as a \mathbb{C} -algebra.

3 Smooth, unramified, and étale morphisms

Let $f : Y \rightarrow X$ be a morphism of schemes. For each morphism $g : X' \rightarrow X$ with X' affine, and each closed subscheme Z of X' defined by a nilpotent ideal of $\mathcal{O}(X')$, we have a canonical map

$$\mathrm{Hom}_X(X', Y) \rightarrow \mathrm{Hom}_X(Z, Y).$$

If this map is always injective/surjective/bijective, we say that f is *formally unramified/smooth/étale*. We drop the “formally” if f is also locally of finite presentation. These properties have all the expected behaviors (local on the base, stable under base change, descendable down faithfully flat quasicompact morphisms).

The definition above is given in terms of an *infinitesimal lifting property*. There are more practical characterizations in terms of differentials; some of these will be exercises. (See EGA IV, part 4, section 17.)

Proposition. *The morphism f is formally unramified if and only if $\Omega_{Y/X} = 0$.*

Proposition. *If f is locally of finite presentation, then f is étale if and only if f is flat and unramified.*

Proposition. *If f is locally of finite presentation, then f is smooth if and only if f is flat and for each $x \in X$, the fibre $f^{-1}(x)$ is geometrically regular over $\kappa(x)$. (That is, for k an algebraic closure of $\kappa(x)$, $f^{-1}(x) \times_{\mathrm{Spec} \kappa(x)} k$ is regular.)*

For example, the projective space \mathbb{P}_X^n is smooth over X .

The difference between regular and geometrically regular shows up only when the field $\kappa(x)$ is imperfect. For instance, put $\kappa = \mathbb{F}_p(x)$, $X = \mathrm{Spec} \kappa$ and $Y = \mathrm{Spec} \mathbb{F}_p(x^{1/p}) = \mathrm{Spec} \kappa[y]/(y^p - x)$. Then Y is a regular scheme, but its base change to an algebraic closure k of κ is

$$\mathrm{Spec} k[y]/(y^p - x) = \mathrm{Spec} k[y]/((y - x^{1/p})^p),$$

which is not regular. For a slightly less trivial example, see Hartshorne exercise III.10.1.

The notion of an étale morphism is an algebro-geometric analogue of the concept of a *covering space* in topology. As such, it forms the basis for one of the most successful notions of cohomology in algebraic geometry, that of *étale cohomology*. I probably won't have time to say more than a few words about that at the end of the course.