18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Divisors, linear systems, and projective embeddings (updated 1 Apr 09)

We conclude the first half of the course by translating into the language of schemes some classical notions related to the concept of a *divisor*. This will serve to explain (in part) why we will be interested in the cohomology of quasicoherent sheaves.

In order to facilitate giving examples, I will mostly restrict to *locally noetherian* schemes. See Hartshorne II.6 for divisors, and IV.1 for Riemann-Roch.

1 Weil divisors

Introduce Hartshorne's hypothesis (*): let X be a scheme which is noetherian, integral, separated, and *regular in codimension 1*. The latter means that for each point $x \in X$ whose local ring $\mathcal{O}_{X,x}$ has Krull dimension 1, that local ring must be regular.

Lemma. *Let A be a noetherian local ring of dimension 1. Then the following are equivalent.*

(a) A is regular.

(b) A is normal.

(c) A is a discrete valuation ring.

(This is why normalizing a one-dimensional noetherian ring produces a regular ring.)

Warning: for a noetherian integral domain, normal implies regular in codimension 1 but not conversely. You have to add Serre's condition S2: for $a \in A$, every associated prime of the principal ideal (a) has codimension 1 when a is not a zerodivisor, and has codimension 0 when $a = 0$.

A *prime (Weil) divisor* on X is a closed integral (irreducible and reduced) subscheme of codimension 1. A formal $\mathbb{Z}$-linear combination of prime divisors is called a *Weil divisor*. If only nonnegative coefficients are used, we say the divisor is *effective*.

For example, let $K(X)$ be the *function field* of X, i.e., the local ring of X at its generic point. (This equals $\mathrm{Frac}(\mathcal{O}(U))$ for any nonempty open affine subscheme U of X.) For $f \in K(X)$ nonzero, we can define a *principal divisor* associated to f as follows. For each prime divisor Z on X, let η_Z be the generic point of Z. Then $\mathcal{O}_{X,\eta_Z}$ is a discrete valuation ring; let v_Z be the valuation. Now define the divisor

$$(f) = \sum_Z v_Z(f) Z;$$

this makes sense because only finitely many $v_Z(f)$ are nonzero. (That's because f restricts to an invertible regular function on some nonempty open subscheme U of X, and $v_Z(f) = 0$ whenever $Z \not\subseteq X - U$.)

Let $\mathrm{Div}\, X$ be the group of Weil divisors of X. The principal divisors form a subgroup (since $(f) + (g) = (fg)$); the quotient by this subgroup is called the *divisor class group* of

X, denoted $\mathrm{Cl}\, X$. For example, if $X = \mathrm{Spec}(A)$ with A a Dedekind domain, then $\mathrm{Div}\, X$ is the group of fractional ideals, and $\mathrm{Cl}\, X$ is the ideal class group. We say two divisors which differ by a principal divisor are *linearly equivalent*.

There are a number of examples in Hartshorne. One of my favorites is that of an *elliptic curve*; here is a summary. Let k be an algebraically closed field (for starters). Let $P(x, y, z) \in k[x, y, z]$ be a homogeneous polynomial of degree 3 defining a nonsingular subvariety C of $\mathbb{P}^2_k$. Pick a point $O \in C(k)$. There is a surjective map $\mathrm{Div}\, X \to \mathbb{Z}$ mapping each prime divisor P to 1, called the *degree*. This map factors through $\mathrm{Cl}\, X$ because each principal divisor has degree 0. The kernel of the degree map $\mathrm{Cl}\, X \to \mathbb{Z}$ is generated by $(P) - (O)$ for $P \in C(k)$. In fact it is *equal* to the set of such elements: given $P, Q \in C$, we first draw the line through P, Q in $\mathbb{P}^2_k$ and find its third intersection point R with C. We then draw the line through R and O in $\mathbb{P}^2_k$ and find its third intersection point S with C. Then

$$(P) + (Q) + (R) \sim (R) + (S) + (O),$$

so

$$(P) - (O) + (Q) - (O) \sim (S) - (O).$$

2 Cartier divisors

When the scheme X is not regular, there is a more restrictive notion of divisors that turns out to be more useful in many cases.

Let $\mathcal{K}$ be the locally constant sheaf associated to the function field $K(X)$. A *Cartier divisor* on X is a section of the sheaf $K(X)/\mathcal{O}^\times$. Using the construction of principal divisors, we obtain a map from Cartier divisors to Weil divisors: if the Cartier divisor is represented on some open subset U of X by the rational function $f \in K(X)$, then the Weil divisor we get should agree with (f) when restricted to U (i.e., only keep the components of those prime divisors meeting U). This map is injective if X is normal, because an integrally closed noetherian domain is the intersections of its localizations at *minimal* prime ideals.

Proposition (Hartshorne, Proposition II.6.11)**.** *Suppose X is locally factorial (i.e., each local ring $\mathcal{O}_{X,x}$ is a unique factorization domain). Then the previous map is an isomorphism. (In particular, this holds if X is regular, because a regular local ring is factorial by a not-so-easy theorem of commutative algebra.)*

Example: if $X = \mathrm{Spec}\, k[x, y, z]/(xy - z^2)$, the ideal (x, z) defines a Weil divisor which is not a Cartier divisor.

Again, there is an obvious notion of a *principal Cartier divisor*, namely one defined by a single element of $K(X)$. The group of Cartier divisors modulo principal divisors is called the *Cartier class group* of X, denoted $\mathrm{CaCl}\, X$.

3 The Picard group

The Cartier class group is "usually" the same as the *Picard group*, namely the group of invertible sheaves on X under the tensor product. Namely, if D is a Cartier divisor on X, let $\mathcal{L}(D)$ be the subsheaf of $\mathcal{K}$ such that

$$\mathcal{L}(D)(U) = \{f \in K(X) : ((f) + (D))|_U \geq 0\}.$$

Assuming that X is normal, this is locally free of rank 1, hence an invertible sheaf. This gives a homomorphism from Cartier divisors to the Picard group, which we see kills the principal divisors. The resulting homomorphism is always injective, even without any hypotheses on X (Hartshorne, Corollary II.6.14) but may not be surjective; however, it is surjective if X is integral (Hartshorne, Proposition II.6.15).

Note that if D is effective, then the function 1 defines a global section of $\mathcal{L}(D)$. Since $\mathcal{L}$ is locally principal, we can locally identify $\mathcal{L}$ with $\mathcal{O}_X$; when we do so, the subsheaf of $\mathcal{L}(D)$ generated by 1 goes into correspondence with an ideal sheaf of $\mathcal{O}_X$, which doesn't depend on any choices. This ideal sheaf defines D as a closed subscheme. In other words, D is the zero locus of a certain section of $\mathcal{L}(D)$. More generally, even if D is effective, we can view D as the zero locus of a *meromorphic section* of $\mathcal{L}(D)$ (meaning a zero locus of $\mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{K}_X$), and indeed the zero locus of any meromorphic section of $\mathcal{L}(D)$ is linearly equivalent to D.

4 Linear systems

Suppose X is an integral separated scheme of finite type over a field k (which need not be algebraically closed). Let $\mathcal{L}$ be an invertible sheaf on X. A *linear system* defined by $\mathcal{L}$ is the set of zero loci of some k-linear subspace H of $H^0(X, \mathcal{L})$. If we take the entire space, that is called the *complete linear system* defined by $\mathcal{L}$.

We can attempt to use the elements of H to define a map $X \to \mathbb{P}^n_k$, where $n = \dim_k(H) - 1$. This might fail to give a morphism because H may have a *base point*, i.e., a point in the intersection of all of the divisors in the linear system. In fact, we get a morphism $X \to \mathbb{P}^n_k$ if and only if H has no base points.

Suppose now that k is algebraically closed, and that X is one-dimensional, projective, irreducible, and nonsingular (i.e., a "curve"). Consider the complete linear system associated to $\mathcal{L}(D)$ for some divisor D.

(a) We get a map $X \to \mathbb{P}^n_k$ if and only if for each closed point $x \in X$, we have $\dim_k H^0(X, \mathcal{L}(D - x) = \dim_k H^0(X, \mathcal{L}(D)) - 1$. (In other words, there must be a section of $\mathcal{L}(D)$ not vanishing at x.)

(b) The map in (a) is injective as a map of sets if and only if for each pair of distinct closed points $x, y \in X$, we have $\dim_k H^0(X, \mathcal{L}(D - x - y)) = \dim_k H^0(X, \mathcal{L}(D)) - 2$. (In other words, there must be a section of $\mathcal{L}(D)$ vanishing at x but not at y, and vice versa.)

(c) The map in (b) is a closed immersion if and only if for each closed point $x \in X$, we have $\dim H^0(X, \mathcal{L}(D - 2x)) = \dim_k H^0(X, \mathcal{L}(D)) - 2$. (In other words, there must be a section of $\mathcal{L}(D)$ not vanishing at x, and a section vanishing to exact order 1 at x.)

(Condition (c) is needed to ensure that the tangent space at x embeds into the tangent space at the image of x. See Remark 7.8.2.)

Since we would like to know under what circumstances X embeds into a projective space, we would like to be able to compute at least the dimension of $H^0(X, \mathcal{L}(D))$ for each divisor D. This quest is greatly abetted by the Riemann-Roch theorem, more on which next time.