18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Étale cohomology (updated 13 May 09)

In this lecture, we give a hint of the theory of *étale cohomology*. Standard references: Milne, *Étale Cohomology* (he also has some more accessible lecture notes online at `jmilne.org`); Tamme, *Introduction to Étale Cohomology*; Freitag-Kiehl, *Étale Cohomology and the Weil Conjectures*. You might want to read Hartshorne Appendix C first for an overview. Also, note that there is a "rogue" volume of SGA called *SGA 4 1/2*, written mostly by Deligne after the fact, which gives a surprisingly legible (albeit in French) account of this stuff.

Since this is the last lecture of the course, I would like to take the opportunity to thank you, the participants, for all the hard work you put in on the problem sets, and especially for all your feedback on the notes. If you have further questions about algebraic geometry, from the general to the specific, I would be happy to discuss them!

1 Motivation: the Weil conjectures

Let X be a variety over a finite field $\mathbb{F}_q$. Weil predicted that the *zeta function* of X, defined as an Euler product

$$\zeta_X(T) = \prod_x (1 - T^{\deg(x \to \mathbb{F}_q)})^{-1}$$

over the closed points of X, could always be interpreted as the power series expansion of a rational function of T; this analogizes the analytic continuation of the Riemann zeta function. For instance, for $X = \mathbb{P}^1$,

$$\zeta_X(T) = \frac{1}{(1-T)(1-qT)}.$$

Weil also predicted analogues of the functional equation of the zeta function, and the Riemann hypothesis. For instance, for X an elliptic curve, Hasse proved that

$$\zeta_X(T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}$$

for some $a \in \mathbb{Z}$. This expression has the symmetry property that

$$\zeta_X(q^{-1}/T) = \zeta_X(T).$$

(This example is a bit lucky; more generally, you might be off by a factor of $q^a T^b$ for some $a, b \in \mathbb{Z}$. For X of pure dimension n, you should compare $\zeta_X(T)$ with $\zeta_X(q^{-n}/T)$.) Hasse also proved that

$$|a| \leq 2\sqrt{q},$$

or equivalently, the numerator polynomial $1 - aT + qT^2$ has complex roots of norm $q^{-1/2}$.

Weil also noticed that the degrees of the factors in the zeta function appeared to have topological meaning. Namely, if X is obtained from a smooth proper scheme over some arithmetic ring (i.e., a localization of the ring of integers in a number field) by reduction modulo a prime, then the degrees of the factors in $\zeta_X(T)$ correspond to the Betti numbers of $(X \times \mathbb{C})^{\mathrm{an}}$. For example, the degrees of the factors $1-T, 1-aT+qT^2, 1-qT$ in the elliptic curve case match the Betti numbers $1,2,1$ of a genus 1 Riemann surface.

Weil proved analogues of all these assertions for arbitrary curves, and (based on some evidence from Fermat hypersurfaces) conjectured analogues for higher dimensional varieties. More precisely, he predicted the existence of a cohomology theory $H^i(\cdot)$ for varieties over $\mathbb{F}_q$, taking values in finite dimensional vector spaces over a field K of characteristic zero, in which the number of $\mathbb{F}_q$-rational points (i.e., the fixed points of the q-power Frobenius map) could be computed using an analogue of the Lefschetz fixed point formula in topology:

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2\dim(X)} (-1)^i \operatorname{Trace}(F_q^n, H^i(X)).$$

This immediately implies rationality of $\zeta_X(T)$. Symmetry should follow from a form of Poincaré duality, i.e., a perfect pairing

$$H^i(X) \times H^{2\dim(X)-i}(X) \to H^{2\dim(X)}(X) \to K.$$

The Riemann hypothesis is not quite as purely formal a consequence, since it is basically a nonnegativity condition, whereas K need not have anything to do with $\mathbb{R}$. But never mind that for now.

2 Curves

For curves, Weil proved his conjectures by constructing an algebraic group associated to a curve C, called the *Jacobian variety* $J(C)$. Over $\mathbb{C}$, this gives a complex torus which had been constructed by Abel-Jacobi using abelian integrals.

For a prime ℓ not equal to the characteristic of $\mathbb{F}_q$, and a positive integer n, the group $J(C)(\overline{\mathbb{F}_q})[\ell^n]$ of geometric ℓ^n-torsion points is abstractly isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$, for g the genus of C. The absolute Galois group of $\mathbb{F}_q$ acts by $(\mathbb{Z}/\ell^n\mathbb{Z})$-module endomorphisms. If we take the inverse limit over n, we get a $\mathbb{Z}_\ell$-module $T_\ell(J(C))$ equipped with an action of the absolute Galois group; it is nowadays called the *Tate module* of C. (For instance, if C is an elliptic curve, then $J(C) = C$.)

This gives the H^1 (or really its dual) in a good cohomology theory. The symmetry comes from the Tate pairing. The Riemann hypothesis can be deduced using the Hodge index theorem, which gives a nonnegativity (or really a nonpositivity) assertion for the intersection pairing on $C \times_{\mathbb{F}_q} C$.

Aside: a noncohomological proof, using only Riemann-Roch and some clever estimates, was found later by Stepanov (and simplified by Bombieri). Good reference: Lorenzini's *Invitation to Arithmetic Geometry*.

3 Why étale?

One might think that coherent sheaf cohomology, as we have developed in this course, might be useful against the Weil conjectures. However, it has several problems: it lives in characteristic p rather than characteristic 0 (so it can only aspire to prove rationality mod p, rather than integrally), and its dimensions do not match Betti numbers. For instance, sheaf cohomology on a scheme of dimension n only goes up to index n, rather than $2n$.

Grothendieck realized that one might get around this by trying to make an analogue of topological cohomology in which étale maps play the role of local homeomorphisms. For instance, recall one of the consequences of GAGA: for a smooth proper variety X over $\mathbb{C}$, every finite covering space map comes from a unique finite étale cover of X. Thus the profinite completion of the topological fundamental group can be recovered as an inverse limit of Galois groups of these étale covers.

Perhaps a better justification for considering étale covers is the following. For X a complex analytic variety and $x \in X$, the local ring $\mathcal{O}_{X,x}$, while not complete, is *henselian*: the conclusion of Hensel's lemma still holds. (That is, given a polynomial over $\mathcal{O}_{X,x}$, any simple root of the reduction modulo the maximal ideal lifts uniquely to a root.) This is not true for schemes, though. A related geometric statement is that if $f : Y \to X$ is an étale morphism of schemes, and $x \in X$ is a point, then there is no way to draw disjoint open neighborhoods of the points of $f^{-1}(x)$, so you cannot view the étale map as a local homeomorphism.

4 Topology revisited

In order to combine the ideas about étale covers with sheaf cohomology, Grothendieck had to take the apparently drastic step of modifying the notion of a topology on a space. But in retrospect, this isn't such a strange modification to make. After all, presheaves on a topological space X are nothing more than contravariant functors on the category $\underline{X}$ of open sets. Why not state all the sheaf axioms in terms of the structure of that category?

Grothendieck realized that stating the sheaf axiom really only requires knowing what an open cover is, leading to the following definition. Let $\mathcal{C}$ be a category admitting fibre products. A *Grothendieck topology* consists of the following data. For each $X \in \mathcal{C}$, you must tell me which collections of morphisms $\{U_i \to X\}_{i \in I}$ are *coverings* of X. This prescription must satisfy some hypotheses.

- Any isomorphism $X \to Y$ is by itself a cover of Y.
- For any $Y \to X$, if $\{U_i \to X\}$ is a cover, then $\{U_i \times_X Y \to Y\}$ is a cover. That is, open covers can be restricted to open subsets.
- If $\{U_i \to X\}$ is a cover, and for each i $\{V_{ij} \to U_i\}$ is a cover, then $\{V_{ij} \to X\}$ is also a cover. That is, covering each open in a cover gives a cover.

(Strictly speaking, this is a *Grothendieck pretopology* because it only gives you the analogue of a *basis* for a topology. You should really throw in all coverings "generated" by these too.)

A category equipped with a Grothendieck topology is called a *site*. For instance, the *big étale site* of a scheme S is the category of all S-schemes, in which coverings are collections of étale morphisms which form a set-theoretic cover. That is, $\{U_i \to X\}$ is a cover if and only if each U_i is étale and the union of their images is X. (If you only bother keeping objects which are themselves étale over S, you get the *small étale site*.)

There are many other useful Grothendieck topologies that occur frequently in algebraic geometry. These include the *fppf topology* (fidèlement plat de présentation finie = faithfully flat of finite presentation), the *fpqc topology* (fidèlement plat et quasicompact = faithfully flat quasicompact), the *smooth topology*, the *flat topology*, the *syntomic topology* (flat and locally of finite presentation), the *Nisnevich topology* (étale, but each point must be covered by a point with the *same* residue field), etc. There are also useful examples where you start with a usual topological space but use only some of the available open covers; this occurs in the definition of *rigid analytic spaces* (i.e., analytic spaces over a *nonarchimedean* complete field like $\mathbb{Q}_p$).

Anyway, once you know what a Grothendieck topology is, you can define a *sheaf* of abelian groups (say) on it. Namely, you want a contravariant functor F from your category to $\underline{\mathrm{Ab}}$, such that for any cover $\{U_i \to X\}$, we have an exact sequence

$$0 \to F(X) \to \prod_i F(U_i) \to \prod_{i,j} F(U_i \times_X U_j)$$

where the last map computes a section on $F(U_i \times_X U_j)$ as the restriction from U_i minus the restriction from U_j. For instance, in most reasonable cases, the *structure sheaf* $F(X) = \mathcal{O}_X$ is a sheaf.

There is also a notion of *sheafification* but this is complicated by the fact that we don't have points with with to define stalks. No matter: what are points anyway but decreasing families of open sets? One can make an artificial definition of "points" in that fashion; this brings one dangerously close to the notion of a *topos*, which I will skip over entirely. (Roughly speaking, a topos is the category of sheaves on a site with values in a given category, like sets or abelian groups.)

5 Étale cohomology in practice

We can now define sheaf cohomology on any site with a final object as the derived functors of global sections, meaning sections over the final object. (One can fix this even if there is no final object, by taking a compatible family of sections over *every* element of the site. Yeesh.)

However, it's not so straightforward to compute étale cohomology of a scheme X with coefficients in a sheaf $\mathcal{F}$. On one hand, writing down étale cochains is not a problem: you specify an étale cover of X and then some sections on each element of the cover. Writing down cocycles isn't that much harder: you have to write down another étale cover on which you can check that the differential of your cochain vanishes. The hard part is, given a cochain, how do you tell whether it is zero or not?

Despite this complication, one can prove quite a lot. For instance, if you start with a quasicoherent sheaf $\mathcal{F}$ on a scheme X, you get a sheaf on its big and small étale sites by setting the sections over an open $i : U \to X$ to be $i^*\mathcal{F}$. But this is a boring example, because the resulting sheaf cohomology turns out to agree with usual sheaf cohomology on the "Zariski site" (i.e., what we already know).

What makes the étale site fun is that you get strange new sheaves, much more akin to the locally constant sheaves in topology, and their cohomology is quite interesting. For instance, you can make a locally constant sheaf associated to any (pro)finite abelian group (by sheafifying the constant presheaf), and this gives you something with topological meaning.

Theorem. *Let X be a smooth proper scheme over $\mathbb{C}$. Then for any prime ℓ, the cohomology of the étale locally constant sheaf associated to the ℓ-adic integers $\mathbb{Z}_\ell$ computes the topological Betti numbers of X.*

The fun comes when you start with a scheme over an arithmetic base, like $\mathbb{Q}$. If you extend the base to $\overline{\mathbb{Q}}$ and then take étale cohomology with coefficients in $\mathbb{Z}_\ell$, the result carries an action of the absolute Galois group of $\mathbb{Q}$. E.g., for an elliptic curve, the first étale cohomology is (dual to) the ℓ-adic Tate module, i.e., the inverse limit of the ℓ-power torsion groups viewed as a Galois representation.

6 Back to the Weil conjectures

Let X be a smooth proper scheme over the finite field $\mathbb{F}_q$. Pick any prime $\ell \neq q$. For each positive integer n, we can consider the locally constant étale sheaf $\underline{\mathbb{Z}/\ell^n\mathbb{Z}}_X$ on X. Let $\underline{\mathbb{Z}_\ell}_X$ be the inverse limit of these; this is *not* the same as the locally constant étale sheaf generated by $\mathbb{Z}_\ell$. (E.g., in the example of the elliptic curve, that is because the ℓ^∞-power torsion is not defined over a *finite* extension of the base field.)

Nonetheless, $\underline{\mathbb{Z}_\ell}$ is a good sheaf to work with. (It is an example of a sheaf which is *lisse*, or *smooth* if you prefer to translate from the French.) We will be interested in working with the

$$H^i(X) = H^i_{\mathrm{et}}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \underline{\mathbb{Z}_\ell}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

which is a collection of $\mathbb{Q}_\ell$-vector spaces. These turn out (with some effort) to be finite dimensional over $\mathbb{Q}_\ell$, and carry a Lefschetz trace formula. This proves rationality of the zeta function.

Aside: rationality had already been proved by Dwork around 1960 using p-adic analytic methods, but not using cohomology. Nowadays, though, Dwork's proof has been reinterpreted in terms of a different Weil cohomology, called *rigid cohomology*, taking values in a p-adic field. (Remember that $\ell = p$ is excluded in étale cohomology, because this case behaves badly. For instance, an elliptic curve over an algebraically closed field of characteristic p has at most p points killed by p, not p^2.)

Returning to étale cohomology, there is also a Poincaré duality which implies symmetry. The Riemann hypothesis, of course, is more subtle; Grothendieck had predicted it would follow from a suitable analogue of the Hodge index theorem, which was one of his *standard conjectures*. This analogue is still open; instead, Deligne proved the Riemann hypothesis by a rather clever combination of ideas, including an algebro-geometric variant of the "Rankin squaring" argument from classical modular forms. Laumon later gave a similar but technically simpler proof by adding the use of a *cohomological Fourier transform*. (These proofs are largely independent of which Weil cohomology you are using. In particular, with some effort they can be transposed into rigid cohomology.)