

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
More properties of morphisms (updated 5 Mar 09)

Note that finite presentation is not discussed in EGA 1; see EGA 4.1 instead.

1 More about separated morphisms

Lemma. *The composition of closed immersions is a closed immersion.*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be closed immersions. Since the property of being a closed immersion is local on the base, we may assume $Z = \operatorname{Spec}(A)$ is affine. Then $Y = \operatorname{Spec}(B)$ for B a quotient of A , so $X = \operatorname{Spec}(C)$ for C a quotient of B . Hence C is a quotient of A , proving the claim. (A similar argument shows that a composition of finite morphisms is finite.) \square

Lemma. (a) *Any closed immersion is separated.*

(b) *A composition of separated morphisms is separated.*

(c) *Separatedness is stable under base change.*

(d) *A product of separated morphisms is separated.*

(e) *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms, $g \circ f$ is separated, and g is separated, then f is separated.*

(f) *If $f : X \rightarrow Y$ is separated, then $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ is separated.*

Proof. We know (a) because closed immersions are affine and affine morphisms are separated. We know (c) from the previous handout. Parts (d)-(f) follow once we also have (b); see exercises.

It remains to prove (b). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be separated morphisms. Then $X \times_Y X$ maps to $X \times_Z X$; in fact, this morphism is the base change of the closed immersion $\Delta : Y \rightarrow Y \times_Z Y$ by $f \times f : X \times_Z X \rightarrow Y \times_Z Y$. (To check this: use functor-of-points to reduce to the analogous assertion for sets. This can be checked with Z equal to a singleton set, so we just want to know that for a morphism of sets $X \rightarrow Y$, the fibre product of Y and $X \times X$ over $Y \times Y$ equals $X \times_Y X$. This is obvious.) Hence $X \times_Y X \rightarrow X \times_Z X$ is a closed immersion. Since the composition of closed immersions is a closed immersion (previous lemma), we find that $X \rightarrow X \times_Y X \rightarrow X \times_Z X$ is a closed immersion. \square

2 Quasicompact morphisms

A morphism $f : Y \rightarrow X$ with X affine is *quasicompact* if Y is quasicompact as a topological space. This definition satisfies the strong collater (exercise), so we get a notion which is local on the base and stable under base change.

Exercise. *Any affine morphism is quasicompact.*

3 Finite type and finite presentation

Let A be a ring. Recall that an A -algebra B is *finitely generated* if it is of the form $A[x_1, \dots, x_n]/I$ for some nonnegative integer n and some ideal I of $A[x_1, \dots, x_n]$. If I can be chosen to be a finitely generated ideal, we say that B is *finitely presented*; this is of course automatic if A is noetherian (as it will be in most of our examples).

Let $f : Y \rightarrow X$ be a morphism of schemes with $X = \text{Spec}(A)$ affine. We say f is *locally of finite type/presentation* if Y is a union of open subschemes, each of the form $\text{Spec}(B)$ with B a finitely generated/presented A -algebra. If only finitely many such open subschemes are needed, we say f is *of finite type/presentation*. These satisfy the strong collater (exercise).

If $f : Y \rightarrow X$ is of finite type, we sometimes say that Y is of finite type over X . Similarly for the other definitions.

Obvious: any finite morphism, including any closed immersion, is of finite type.

Exercise. A morphism $f : Y \rightarrow X$ is of finite type/presentation if and only if it is quasi-compact and locally of finite type/presentation.

4 Algebraic varieties

We can now give a scheme-theoretic rendition of the theory of abstract algebraic varieties, in the sense of 18.725. (But see below.)

Let k be an algebraically closed field. An *affine variety* is a locally ringed space defined by some data of the following form. Pick a nonnegative integer n and an ideal I of $k[x_1, \dots, x_n]$, and put $X = V(I)$. Equip X with the Zariski topology, i.e., take a basis of open sets of the form $D(g) = \{x \in X : g(x) \neq 0\}$ for $g \in k[x_1, \dots, x_n]$. Define a *regular function* on an open subset U of X to be a function $h : U \rightarrow k$ such that for each $x \in U$, there exist $f, g \in k[x_1, \dots, x_n]$ and a nonnegative integer m such that g vanishes nowhere on U while $g^m h - f$ vanishes identically on U . Then the regular functions on U form a sheaf.

In the context of schemes, we interpret X to be the set of maximal ideals in $\text{Spec}(A)$ for $A = (k[x_1, \dots, x_n]/I)^{\text{red}}$, equipped with the structure of a locally ringed space given by restriction from $\text{Spec}(A)$.

Now recall that an *abstract algebraic variety* is a locally ringed space covered by affine varieties.

Theorem 1. *The category of abstract algebraic varieties over the algebraically closed field k is equivalent to the category of schemes which are reduced and locally of finite type over $\text{Spec}(k)$.*

Proof. Exercise. The key point is to check that if $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ for A, B two reduced finitely generated k -algebras, then the morphisms from X to Y are the same as the morphisms of the corresponding algebraic varieties. But that is because they both correspond to ring homomorphisms $B \rightarrow A$. \square

Beware that there is no universal definition of *algebraic varieties*, because everyone seems to prefer to add additional hypotheses. For instance, Hartshorne (see Chapter I) forces his varieties to be separated (as often do I). Some authors also force their varieties to be *irreducible*, i.e., not admitting two disjoint open subschemes. And so on.

5 Proper morphisms

We would like to have an algebraic analogue of the notion of a *compact* algebraic variety over the complex numbers. For this, we introduce the notion of properness.

A morphism $f : Y \rightarrow X$ of schemes is *proper* if it is separated, of finite type, and *universally closed*. The latter means that any base change of f is a closed map of topological spaces (i.e., carries closed sets to closed sets); this condition comes from the notion of a proper map of topological spaces (see exercises). Since these properties are all local on the base and stable under base change (the last one by fiat), properness is also.

The definition of properness is rather hard to check. One easy case: a closed immersion is separated (because it's affine), of finite type (obvious), and universally closed (because any base change is still a closed immersion, so has closed image), so is proper. Besides this example, and the following slightly fancier example...

Exercise. *Any finite morphism (including any closed immersion) is proper.*

... all examples of properness will ultimately be extracted from the following theorem.

Theorem 2. *The morphism $f : \mathbb{P}_{\mathbb{Z}}^n \rightarrow \operatorname{Spec} \mathbb{Z}$ is proper.*

Hartshorne proves this using the valuative criterion for properness (under a somewhat mysterious noetherian hypothesis). I'll ultimately prove this following EGA, but I need to wait until the next lecture so I can say more about projective spaces in the interim. I will point out now that the fact that f is of finite type is evident from the glueing construction, and the separatedness may be obtained by describing the diagonal $\Delta : \mathbb{P}_{\mathbb{Z}}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^n \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^n$ explicitly (exercise).

As for separated morphisms, we have some properties.

Lemma. (a) *Any closed immersion is proper.*

(b) *A composition of proper morphisms is proper.*

(c) *Properness is stable under base change.*

(d) *A product of proper morphisms is proper.*

(e) *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms, $g \circ f$ is proper, and g is separated, then f is proper.*

(f) *If $f : X \rightarrow Y$ is proper, then $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ is proper.*

Proof. Again, (d)-(f) follow from (a)-(c). We already observed (a) and (c). To check (b), we already checked that separatedness composes. Finite type composes by an argument similar to the proof that closed immersions compose. Universal closedness composes because a composition of closed maps of topological spaces is again closed. \square

Corollary. *Any morphism $f : X \rightarrow Y$ that factors as a closed immersion of X into $\mathbb{P}_Y^n = \mathbb{P}_{\mathbb{Z}}^n \times_{\mathrm{Spec} \mathbb{Z}} Y$ followed by the projection $\mathbb{P}_Y^n \rightarrow Y$ is proper.*

The converse is not true even over \mathbb{C} , as there are compact algebraic varieties which are not closed subvarieties of any projective space. See the appendices to Hartshorne for an example. One can often deal with these using *Chow's lemma*, about which more later.