18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Higher Riemann-Roch

In this lecture, we discuss some higher-dimensional versions of the Riemann-Roch theorem: the Riemann-Roch theorem for surfaces, the Hirzebruch-Riemann-Roch theorem, and the Grothendieck-Riemann-Roch theorem. For the first, see Hartshorne V.1; for the others, see Chapter 15 of Fulton's *Intersection Theory* (a well-deserved winner of the Steele Prize for mathematical exposition).

1 Surfaces

Let X be a smooth irreducible projective surface over an algebraically closed field k. Let K be a canonical divisor on X. As in the case of curves, the Riemann-Roch theorem combines an input from Serre duality with an Euler characteristic calculation.

The input from Serre duality is that for any divisor D,

$$H^0(X, \mathcal{L}(D)^\vee \otimes \omega_X) \cong H^2(X, \mathcal{L}(D))^\vee.$$

We can thus write the Euler characteristic $\chi(X, \mathcal{L}(D))$ as

$$\dim_k H^0(X, \mathcal{L}(D)) - \dim_k H^1(X, \mathcal{L}(D)) + \dim_k H^0(X, \mathcal{L}(K - D)).$$

Unfortunately, we can't do much with the term $\dim_k H^1(X, \mathcal{L}(D))$ other than give it a name: it's called the *superabundance* of D. However, we do at least know that it is nonnegative, and this turns out to be surprisingly useful.

The Euler characteristic calculation is made as follows. Write D as the difference between two effective divisors $C - E$ with no common components. We then have exact sequences

$$0 \to \mathcal{L}(C - E) \to \mathcal{L}(C) \to \mathcal{L}(C) \otimes \mathcal{O}_E \to 0, \qquad 0 \to \mathcal{O}_X \to \mathcal{L}(C) \to \mathcal{L}(C) \otimes \mathcal{O}_C \to 0.$$

By additivity of χ, we get

$$\chi(X, \mathcal{L}(C - E)) = \chi(X, \mathcal{O}_X) + \chi(C, \mathcal{L}(C)) - \chi(E, \mathcal{L}(C)).$$

The first term we are happy to leave alone since it depends only on X. The other two are calculated using *intersection theory* on the surface X. For instance, the term $\chi(E, \mathcal{L}(C))$ equals $C \cdot E + 1 - g_E$, where g_E is the genus and $C \cdot E$ is the length of the scheme-theoretic intersection $C \times_X E$ (this amounts to Riemann-Roch on the curve E).

The term $\chi(C, \mathcal{L}(C))$ is a bit trickier: it is $C \cdot C + 1 - G_C$ where $C \cdot C = C^2$ is the *self-intersection* of C. That can be defined as $C \cdot C'$ if C is linearly equivalent to a divisor C' having no common components with C, but that is not always possible. In fact, the correct definition is to force the intersection pairing to be bilinear, and this sometimes involves letting C^2 take negative values. For instance, if you blow up P^2 at a point, the exceptional divisor has self-intersection -1. (This is a general pattern; one can in fact blow down any curve isomorphic to $\mathbb{P}^1$ with self-intersection -1.)

Moreover, one can write the genera of C and E in terms of the canonical divisor K, basically using Riemann-Roch again:

$$C \cdot (C + K) = 2g_C - 2, \qquad E \cdot (E + K) = 2g_E - 2.$$

So

$$\chi(X, \mathcal{L}(D)) = \frac{1}{2} D \cdot (D - K) + \chi(X, \mathcal{O}_X).$$

As in the case of curves, this is useful for many calculations involving the geometry of surfaces, such as the Hodge index theorem and the Nakai-Moishezon criterion. These in turn figure in the classification of surfaces (which you should read about in Hartshorne if you are interested in Abhinav's work).

Theorem (Hodge index theorem)**.** *Fix a projective embedding of X, and let H be a divisor with $\mathcal{L}(H) \cong \mathcal{O}_X(1)$. Then for any divisor D such that $D \cdot H = 0$, we have $D^2 \leq 0$. (This also holds if H is ample, i.e., some positive multiple of H comes from an $\mathcal{O}_X(1)$.)*

Theorem (Nakai-Moishezon criterion)**.** *A divisor D on X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C on X.*

2 Hirzebruch's generalization

Hirzebruch noticed that the Euler characteristic aspect of Riemann-Roch could be generalized to handle arbitrary vector bundles on arbitrary smooth varieties over an algebraically closed field k. Let me state his result and then explain what it means.

Theorem (Hirzebruch)**.** *Let X be a smooth proper scheme over k. Let $\mathcal{F}$ be a locally free coherent sheaf on X. Then*

$$\chi(X, \mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(T_X).$$

Here T_X is the *tangent bundle* of X, i.e., the dual to the bundle ω_X of Kähler differentials (which is also called the *cotangent bundle*).

The *Chern character* ch is a certain map from coherent sheaves on X to a certain group of *cycles* on X. The latter are formal $\mathbb{Q}$-linear combinations of subschemes of X modulo a relation of *rational equivalence*. You should imagine this as generalizing the function taking a line bundle $\mathcal{L}$ on a curve C to (the equivalence class of) the divisor of a nonzero rational section of $\mathcal{L}$.

The group of cycles is graded by codimension, and forms a commutative ring under the (appropriately defined) intersection pairing with the identity being the class of X itself in codimension 0. The Chern character is usually split up as $\sum_d c_d(\cdot)$ with c_d being the bit in codimension d; for $\mathcal{F}$ locally free of rank 1, we always have

$$c_d(\mathcal{F}) = \frac{1}{d!} c_1(\mathcal{F})^d.$$

The *Todd class* td is another such map on coherent sheaves, which I won't try to construct here, except to give the characterizing identity: for $\mathcal{F}$ locally free of rank d,

$$\operatorname{td}(\mathcal{F}) \cdot \sum_{p=0}^{d} (-1)^p \operatorname{ch}(\wedge^p \mathcal{F}^\vee) = c_d(\mathcal{F}).$$

. The point is that it depends only on X, not on $\mathcal{F}$.

The Chern character and the Todd class are both examples of *characteristic classes* of vector bundles, which originally appeared in algebraic topology as tools for classifying manifolds. For instance, Milnor uses them to construct differentiable manifolds which are homeomorphic but not diffeomorphic to the 7-sphere, the so-called *exotic 7-spheres*. See Milnor and Stasheff, *Characteristic Classes* for an introduction.

Oh, and $\int_X$ means use intersection theory (which is a pretty complicated thing to define, as evidenced by the length of Fulton's book), keep only the zero-dimensional part, and count points.

3 Grothendieck's generalization

In characteristic fashion, Grothendieck noticed that one can make a relative version of the Hirzebruch-Riemann-Roch theorem. Also, one can drop the locally free condition.

Theorem (Grothendieck)**.** *Let $f : X \to Y$ be a proper morphism of smooth schemes over an algebraically closed field k. Then for any coherent sheaf $\mathcal{F}$ on X,*

$$\operatorname{ch}(f_*\mathcal{F}) \cdot \operatorname{td}(T_Y) = f_*(\operatorname{ch}(\mathcal{F}) \cdot \operatorname{td}(T_X)).$$

One has to define direct image for cycles; I won't try here.

It should be noted that already our formulation of Hirzebruch's statement is Grothendieck's; the original statement was made in the language of topology. One byproduct of this work is the development of *K-theory*, which is now a frequently occurring construction in both algebraic topology and algebraic geometry.