18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Hilbert polynomials and flatness (revised 17 Apr 09)

See Hartshorne III.9 again.

1 Hilbert polynomials

Let k be a field (not necessarily algebraically closed). Let $j : X \to \mathbb{P}^r_k$ be a closed immersion for some $r \geq 1$. Write $\mathcal{O}_X(1)$ for the inverse image by j of the twisting sheaf $\mathcal{O}(1)$. Let $\mathcal{F}$ be a finitely generated quasicoherent sheaf on X.

The *Euler characteristic* of $\mathcal{F}$ is defined as

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F});$$

we know from Serre's finiteness theorem that each summand is finite, and we also know that there are no terms in dimension greater than r. So this is indeed a well-defined integer.

Lemma. *The Euler characteristic is additive in short exact sequences; that is, if*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is exact, then

$$\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H}).$$

Proof. Exercise in the long exact sequence in cohomology. $\square$

Corollary. *If*

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of finitely generated quasicoherent sheaves, then

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

Theorem. *There exists a polynomial $P(z) \in \mathbb{Q}[z]$ such that*

$$\chi(X, \mathcal{F}(n)) = P(n) \qquad (n \in \mathbb{Z}).$$

Moreover, the degree of P is at most the dimension of X.

Proof. By replacing $\mathcal{F}$ by $j_*\mathcal{F}$, we may reduce to the case $X = \mathbb{P}^r_k$. Also, changing the base field doesn't change any of the dimensions (e.g., by looking at Čech cohomology; this is a special case of the *flat base change theorem*), so we may assume k is algebraically closed.

We proceed by induction on the dimension of the support of $\mathcal{F}$. If that support is empty (i.e., $\mathcal{F}$ is the zero sheaf), then obviously $P(z) = 0$ works.

Otherwise, form an exact sequence

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \stackrel{\times x_r}{\to} \mathcal{F} \to \mathcal{H} \to 0$$

and note that $\mathcal{G}$ and $\mathcal{H}$ have support of lower dimension than $\mathcal{F}$ provided that we ensure that the hyperplane $x_r = 0$ does not contain any component of the support of $\mathcal{F}$. (We can arrange this given that k is algebraically closed; see exercises. In fact, k infinite would be sufficient.) By the induction hypothesis, we know that $\chi(\mathbb{P}^r_k, \mathcal{F}(n)) - \chi(\mathbb{P}^r_k, \mathcal{F}(n-1))$ is a polynomial in n of degree at most $\dim(\mathbf{Supp}\,\mathcal{F}) - 1$. It is an elementary exercise in algebra to then see that $\chi(\mathbb{P}^r_k, \mathcal{F}(n))$ is a polynomial in n of degree at most $\dim(\mathbf{Supp}\,\mathcal{F})$. $\square$

The polynomial $P(n)$ is called the *Hilbert polynomial* of the sheaf $\mathcal{F}$; in case $\mathcal{F} = \mathcal{O}_X$, we call it the *Hilbert polynomial* of the scheme X itself. Note that by Serre's vanishing theorem, for some n_0, we have

$$P(n) = \dim_k H^0(X, \mathcal{F}) \qquad (n \geq n_0);$$

this was the original definition of the Hilbert polynomial.

For example, the Hilbert polynomial of $\mathbb{P}^r_k$ itself is $P(n) = \binom{n+r}{n}$. For another example, the Hilbert polynomial of the subscheme $\operatorname{Spec} k[x]/(x^2)$ of $\mathbb{P}^1_k$ is $P(n) = 2$, which is the same as the Hilbert polynomial of a scheme consisting of two distinct reduced points. This is suggestive, because this scheme can indeed be written as a flat limit of pairs of distinct points.

2 Flatness and Hilbert polynomials

The Hilbert polynomial can be used to give the following numerical criterion for flatness. (The locally noetherian hypothesis is important; I think one can replace "integral" by "connected and reduced".)

Theorem. *Let T be an integral (locally) noetherian scheme. Let $X \subseteq \mathbb{P}^r_T$ be a closed subscheme. Let $\mathcal{F}$ be a coherent sheaf on X. For each $t \in T$, let $P_t \in \mathbb{Q}[z]$ be the Hilbert polynomial of the pullback of $\mathcal{F}$ to the fibre X_t viewed as a subscheme of $\mathbb{P}^r_{\kappa(t)}$ (where $\kappa(t) = \mathcal{O}_{T,t}/\mathfrak{m}_{T,t}$ is the residue field of the point t). Then $\mathcal{F}$ is flat relative to $X \to T$ if and only if P_t is constant as a function of t.*

In particular, X itself is flat over T if and only if the Hilbert polynomial of X_t is constant as t varies. This gives us a way to check whether a morphism is flat which we were sorely lacking before.

Proof. (Compare Hartshorne Theorem III.9.9, or EGA III §7.9.) We first note that we can reduce to the case $X = \mathbb{P}^r_T$ by replacing $\mathcal{F}$ with its direct image. We next note that it suffices to consider the case where $T = \operatorname{Spec} A$ for A a local integral noetherian ring.

We then show that $\mathcal{F}$ is flat over T if and only if $H^0(X, \mathcal{F}(m))$ is finite free over A for m sufficiently large. On one hand, if $\mathcal{F}$ is flat over T, then so are all the terms in the sheafy

Čech resolution of $\mathcal{F}(m)$ for the usual open cover $\mathfrak{U}$ (since open immersions are flat). Taking global sections, we see that the terms of the exact sequence

$$0 \to H^0(X, \mathcal{F}(m)) \to \check{C}^0(\mathfrak{U}, \mathcal{F}(m)) \to \cdots \to \check{C}^r(\mathfrak{U}, \mathcal{F}(m)) \to 0$$

are all flat except possibly for the first term. This then implies flatness of $H^0(X, \mathcal{F}(m))$ (exercise). Since it's also finitely generated over A by Serre's finiteness theorem, it is free.

On the other hand, if we pick m_0 such that $H^0(X, \mathcal{F}(m))$ is finite free over A for $m \geq m_0$, then we can recover $\mathcal{F}$ as $\tilde{M}$ for

$$M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m)).$$

Since M is flat, so is $\mathcal{F}$.

We now need to show that $H^0(X, \mathcal{F}(m))$ is finite free for m large if and only if P_t is constant. I claim that this follows by checking

$$H^0(X_t, \mathcal{F}_t(m)) = H^0(X, \mathcal{F}(m)) \otimes_A \kappa(t)$$

for m large (even if I don't prove this uniformly in t). Namely, if $H^0(X, \mathcal{F}(m))$ is finite free over A for $m \geq m_0$, then for each t, for m large, I find that P_t equals P_η for η the generic point of T. On the other hand, if P_t is the same for the generic point and the closed point, then I can make m large enough to work for both, and obtain finite freeness of $H^0(X, \mathcal{F}(m))$.

To check

$$H^0(X_t, \mathcal{F}_t(m)) = H^0(X, \mathcal{F}(m)) \otimes_A \kappa(t),$$

we may reduce to the case where t is the closed point by replacing A with $\mathcal{O}_{T,t}$. Since A is noetherian, we can find a short exact sequence

$$A^{\oplus n} \to A \to \kappa(t) \to 0$$

of A-modules. We can then tensor with $\mathcal{F}$ to get an exact sequence; it follows (exercise) that

$$H^0(X, \mathcal{F}(m)^{\oplus n}) \to H^0(X, \mathcal{F}(m)) \to H^0(X, \mathcal{F}_t(m)) \to 0$$

is exact for m large. I can pull out the direct sum, and then we read off what we want. $\square$

3 Hilbert schemes

It turns out that there is a universal family of closed subschemes of projective space with a fixed Hilbert polynomial.

Theorem (Grothendieck)**.** *Fix a field k and an integer r. Let $P(z) \in \mathbb{Q}[z]$ be a polynomial. There exists a noetherian scheme H over* $\operatorname{Spec} k$ *and a closed subscheme X of $\mathbb{P}^r_H$ which is flat with Hilbert polynomial $P(z)$, such that for any noetherian scheme T and any closed subscheme Y of $\mathbb{P}^r_T$ which is flat with Hilbert polynomial $P(z)$, there is a unique morphism $T \to H$ such that $Y = X \times_H T$ as closed subschemes of $\mathbb{P}^r_T \cong \mathbb{P}^r_H \times_H T$.*

For instance, one can show that a closed subscheme of $\mathbb{P}^r_k$ is a d-dimensional plane if and only if it has Hilbert polynomial $P(n) = \binom{n+d-1}{n}$. The parameter scheme in this case is the *Grassmannian* of d-dimensional planes in $\mathbb{P}^r_k$.

4 Hilbert polynomials, degree, and dimension

Some of the basic information contained in the Hilbert polynomial of a scheme is the following.

Lemma. *Let $P(z)$ be the Hilbert polynomial of a closed subscheme X of $\mathbb{P}^n_k$.*

(a) We have $\deg(P) = \dim(X)$*.*

(b) Put $d = \dim(X)$*. For any d-dimensional plane $P \subseteq \mathbb{P}^n_k$ such that* $\dim(X \cap P) = 0$*, the length of $X \cap P$ is $d!$ times the leading coefficient of P. (This length is called the* degree *of X.)*

Proof. We may assume k is algebraically closed. We first need to know that for a *generic* d-dimensional plane P (i.e., one chosen outside some closed subscheme of the Grassmannian), we have $\dim(X \cap P) = 0$. This follows from the fact that as long as $X \neq \emptyset$, for a generic hyperplane H, we have $\dim(X \cap H) < \dim(X)$ (exercise).

Put $\mathcal{F} = j_* \mathcal{O}_X$ for $j : X \to \mathbb{P}^n_k$ the given closed immersion. For H a hyperplane with $\dim(X \cap H) < \dim(X)$, we have an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the direct image of the structure sheaf of $X \cap H$. If $P(z)$ is the Hilbert polynomial of X, it follows that the Hilbert polynomial of $X \cap H$ is $P(z) - P(z-1)$. From this, both claims follows. ∎