18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Homological algebra (updated 8 Apr 09)

We now enter the second part of the course, in which we use cohomological methods to gain further insight into the theory of schemes. To start with, let us recall some of the basics of homological algebra. The original reference for derived functors is the book *Homological Algebra* of Cartan and Eilenberg, and for cohomological functors is Grothendieck's article *Sur quelques points d'algèbre homologique*; however, any good modern book on homological algebra (e.g., Weibel, *An Introduction to Homological Algebra*) should suffice. (It is worth keeping in mind Lang's suggested exercise in homological algebra: take any book on homological algebra, read the statements of the theorems, and prove them all yourself.)

1 Abelian categories

We saw once before the notion of an *abelian category*. This is a category $\mathcal{C}$ in which each homset has the structure of an abelian group in a manner compatible with composition, with some additional restrictions designed to make things well-behaved. Let's recall some of these. First of all, there must exist *biproducts*, i.e., for any nonnegative integer n and any objects $X_1, \dots, X_n$ in $\mathcal{C}$, there must exist an object Y and morphisms $\iota_i : X_i \to Y$ and $\pi_i : Y \to X_i$ for $i = 1, \dots, n$ such that Y is the product of the X_i (using the π_i) and the coproduct of the X_i (using the ι_i), and $\sum_{i=1}^n \iota_i \circ \pi_i = 1$.

Also, each morphism must have a *kernel* and a *cokernel*. A *kernel* of the morphism $f : X \to Y$ to be a limit of the diagram

$$\begin{tikzcd} X \arrow[dr] & & 0 \arrow[dl] \\ & Y & \end{tikzcd}$$

We write $\mathrm{Ker}(f)$ for the domain of a kernel. Similarly, a *cokernel* of f is a colimit of

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y & & 0 \end{tikzcd}$$

We write $\mathrm{Coker}(f)$ for the codomain of a cokernel.

Finally, we insist that every monomorphsm be the kernel of its cokernel, and every epimorphism be the cokernel of its kernel.

Examples:

1. Ab, the category of abelian groups.

2. $\underline{\mathrm{Mod}}_R$, the category of modules over a ring. We can drop our running commutativity hypothesis if we choose to work with, say, left modules.

3. The category of sheaves on a fixed topological space with values in another abelian category.

I recommend just thinking about the case of abelian groups. The *Freyd-Mitchell embedding theorem* implies that most things you prove about an abelian category can be deduced from the case of abelian groups, where you can use "diagram-chasing" arguments.

2 Complexes and exact sequences

Throughout this section, all objects are in a particular abelian category $\mathcal{C}$.

A sequence of morphisms

$$\cdots \to C^{i-1} \stackrel{d^{i-1}}{\to} C^i \stackrel{d^i}{\to} C^{i+1} \to \cdots$$

is a *complex* if the composition of any two of the arrows is zero, i.e., $d^i \circ d^{i-1} = 0$ for all i. Note that I number the objects so that the arrows point in the increasing direction; this is called a *cohomological grading*. If I numbered things the other way, I would have a *homological grading*. I will mostly talk about the cohomological grading because that is what is most convenient for algebraic geometry. (In a homological grading, you usually write with subscripts instead of superscripts, i.e., $d_i : C_i \to C_{i-1}$.)

The *i-th cohomology* of a complex $C^\cdot$, denoted $h^i(C^\cdot)$, is defined as

$$h^i(C^\cdot) = \frac{\ker(d^i)}{\operatorname{im}(d^{i-1})}.$$

We say that $C^\cdot$ is *exact* if $h^i(C^\cdot) = 0$ for all i.

A *morphism* of complexes $f^\cdot : C^\cdot \to D^\cdot$ is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"] & C^i \arrow[r, "d^i"] \arrow[d, "f^i"] & C^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & D^{i-1} \arrow[r, "d^{i-1}"] & D^i \arrow[r, "d^i"] & D^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

With this definition, we obtain a category of complexes with values in $\mathcal{C}$; this is again an abelian category (exercise).

Any morphism $f^\cdot : C^\cdot \to D^\cdot$ induces maps

$$f^i : h^i(C^\cdot) \to h^i(D^\cdot)$$

for each i. We say f is a *quasi-isomorphism* (or *quasiisomorphism*, but I'll spare you the doubled vowel) if each f^i is an isomorphism; for example, this occurs if f is *homotopic* to the zero map in the following sense. Given two maps $f^\cdot, g^\cdot : C^\cdot \to D^\cdot$, we say that f and g are *homotopic* if there exist a sequence of maps

$$k^i : C^i \to D^{i-1}$$

such that

$$k^{i+1} \circ d^i + d^{i-1} \circ k^i = f - g;$$

this is obviously an equivalence relation. It is an exercise to show that this implies that f and g induce the same maps $h^i(C^\cdot) \to h^i(D^\cdot)$. (The collection of maps k^i are called a *chain homotopy* between f and g.) Important: the fact that a morphism is a quasi-isomorphism is *not* stable under applying functors, but the fact that two morphisms are homotopic is stable under applying functors because it is arrow-theoretic. (This should remind you of the fact that a sequence being exact is not stable under applying functors, but it being a complex is stable.)

The homology functors don't quite capture as much information as possible, just as passing from a filtered object to its associated graded object loses information. A better construction is that of the *derived category* of complexes with values in $\mathcal{C}$; in this construction, one formally inverts all quasi-isomorphisms. This is not completely straightforward, and I won't talk about it more just now.

3 The long exact sequence in cohomology

Let

$$0 \to C^\cdot \to D^\cdot \to E^\cdot \to 0$$

be a short exact sequence of complexes, i.e., a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C^{i-1} \arrow[r] \arrow[d, "d^{i-1}"] & D^{i-1} \arrow[r] \arrow[d, "d^{i-1}"] & E^{i-1} \arrow[r] \arrow[d, "d^{i-1}"] & 0 \\
0 \arrow[r] & C^{i} \arrow[r] \arrow[d, "d^{i}"] & D^{i} \arrow[r] \arrow[d, "d^{i}"] & E^{i} \arrow[r] \arrow[d, "d^{i}"] & 0 \\
0 \arrow[r] & C^{i+1} \arrow[r] \arrow[d] & D^{i+1} \arrow[r] \arrow[d] & E^{i+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

in which the rows are exact, and the columns are complexes. As was shown in a previous exercise, this leads to a long exact sequence

$$\cdots \to h^{i-1}(C^\cdot) \to h^{i-1}(D^\cdot) \to h^{i-1}(E^\cdot) \stackrel{\delta^{i-1}}{\to} h^i(C^\cdot) \to h^i(D^\cdot) \to h^i(E^\cdot) \to \cdots$$

in which the maps $h^\cdot(C^\cdot) \to h^\cdot(D^\cdot)$ and $h^\cdot(D^\cdot) \to h^{i-1}(E^\cdot)$ are the obvious induced ones, and the maps δ^i are the *connecting homomorphisms*. (Recall the definition of δ^i: given an element x in E^{i-1} representing a class in $h^{i-1}(E^\cdot)$, use exactness in the row to lift x to $y \in D^{i-1}$. Then the image of $d^{i-1}(y)$ in E^i equals $d^{i-1}(x) = 0$, so $d^{i-1}(y)$ lifts to $z \in C^i$. The image of $d^i(z)$ in D^{i+1} equals $d^i(d^{i-1}(y)) = 0$, so z represents a class in $h^i(C^\cdot)$. The fact that this class is well-defined independent of choices, and that the resulting map δ^i makes the long sequence exact, were part of the earlier exercise.)

4 Cohomological functors

Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be an additive covariant functor between abelian categories. Recall that F is *left exact* if for any exact sequence

$$0 \to A_1 \to A_2 \to A_3$$

the sequence

$$0 \to F(A_1) \to F(A_2) \to F(A_3)$$

is exact. The functor is *right exact* if for any exact sequence

$$A_1 \to A_2 \to A_3 \to 0$$

the sequence

$$F(A_1) \to F(A_2) \to F(A_3) \to 0$$

is exact. The functor is *exact* if it is both left exact and right exact; equivalently, for any exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

the sequence

$$0 \to F(A_1) \to F(A_2) \to F(A_3) \to 0$$

is exact. This implies that F preserves exact sequences of any length.

Many interesting functors in mathematics are left or right exact but not exact. For example, for $\mathcal{C}$ an abelian category and X an object, the functor $\mathrm{Hom}(X, \cdot)$ carrying Y to $\mathrm{Hom}(X, Y)$ is left exact. (We saw this previously for $\underline{\mathrm{Mod}}_R$ but it holds in general.) We would like to be able to quantify the failure of a functor to be exact; our ability to do this is aided by the presence of objects on which the functor behaves well. For instance, in $\underline{\mathrm{Mod}}_R$, the functor $X \otimes_R \cdot$ behaves badly on a general exact sequence. However, if

$$0 \to Y_1 \to Y_2 \to Y_3 \to 0$$

is a short exact sequence in which Y_3 is a *flat* R-module, then it can be shown that

$$0 \to X \otimes Y_1 \to X \otimes Y_2 \to X \otimes Y_3 \to 0$$

is again exact. For instance, this holds if Y_3 is a *free* R-module.

Assume now that F is a left exact functor. The idea now is to replace the single bad object X first with the complex $0 \to X \to 0 \to \cdots$, then with a quasi-isomorphic complex

$$0 \to X^0 \to X^1 \to \cdots$$

of good objects. If we can lift short exact sequences of maps to short exact sequences of these *resolving complexes*, we can then use the long exact sequence in cohomology to quantify the failure of right exactness. Namely, our short exact sequence

$$0 \to A \to B \to C \to 0$$

will be replaced by a short exact sequence of complexes

$$0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0.$$

If we have chosen the good objects well, then

$$0 \to F(A^{\cdot}) \to F(B^{\cdot}) \to F(C^{\cdot}) \to 0$$

will still form a short exact sequence of complexes, and its long exact sequence in homology

$$0 \to h^0(F(A^{\cdot})) \to h^0(F(B^{\cdot})) \to h^0(F(C^{\cdot})) \xrightarrow{\delta^0} h^1(F(A^{\cdot})) \cdots$$

will tell us something useful. What we really want is that $h^0(F(A^{\cdot})) = A$ and so forth, so that this long exact sequence fills in the gap left at the right end of the exact sequence

$$0 \to F(A) \to F(B) \to F(C).$$

To quantify this notion, we define a *cohomological functor* (or δ*-functor*) between abelian categories $\mathcal{C}_1$ and $\mathcal{C}_2$ to be a sequence of functors

$$T^i : \mathcal{C}_1 \to \mathcal{C}_2 \qquad (i = 0, 1, \dots)$$

plus for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}_1$ a morphism $\delta^i : T^i(C) \to T^{i+1}(A)$ functorial in the sequence (I'll let you draw the diagram), such that the sequence

$$0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta^0} T^1(A) \to T^1(B) \to T^1(C) \xrightarrow{\delta^1} T^2(A) \to \cdots$$

is exact. A cohomological functor is *universal* if given any other cohomological functor U and a natural transformation $f^0 : T^0 \to U^0$, there is a unique sequence of natural transformations $f^i : T^i \to U^i$ starting with f^0 which commute with the δ^i. Given T^0, any two extensions of it to a universal cohomological functor are naturally isomorphic.

This notion does not become useful without a criterion for checking whether a cohomological functor is universal. Here is one. A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between abelian categories is *effaceable* if for any $A \in \mathcal{C}_1$, there is a monomorphism $u : A \to B$ with $F(u) = 0$. I like to think of this in the following way. Most of the time, we deal with functors which are kind of "monotonic", in the sense that under some appropriate hypothesis, the bigger the input object into the functor, the bigger the output object. Effaceable functors are quite the opposite!

Theorem (Grothendieck)**.** *Let $T^\cdot : \mathcal{C}_1 \to \mathcal{C}_2$ be a cohomological functor such that T^i is effaceable for each $i > 0$. Then T is universal.*

Proof. Here's how to construct the natural transformation from T^i to U^i. Given an object A and an index $i > 0$ such that we know the existence and uniqueness of the natural transformation for indices less than i, choose a monomorphism $u : A \to B$ with $T^i(u) = 0$. Then form the long exact sequence $0 \to A \to B \to C \to 0$, apply both cohomological functors, and use the equality $u = 0$ to truncate the upper one:

$$\begin{tikzcd}
T^{i-1}(A) \arrow[r] \arrow[d] & T^{i-1}(B) \arrow[r] \arrow[d] & T^{i-1}(C) \arrow[r, "\delta^{i-1}"] \arrow[d] & T^i(A) \arrow[r] \arrow[d, dashed, "?"] & 0 \\
U^{i-1}(A) \arrow[r] & U^{i-1}(B) \arrow[r] & U^{i-1}(C) \arrow[r, "\delta^{i-1}"] & U^i(A) &
\end{tikzcd}$$

An easy diagram chase shows that there is a unique arrow $T^i(A) \to U^i(A)$ making the diagram commute. It remains to check that:

- the arrow $T^i(A) \to U^i(A)$ does not depend on the choice of u;
- these arrows form a natural transformation.

We leave these verifications as an exercise. $\square$

A typical case is when each object $A \in \mathcal{C}_1$ admits a monomorphism $u : A \to B$ in which B is *acyclic* for T, that is, $T^i(B) = 0$ for $i > 0$. These objects are good in the sense considered above.

Theorem (Acyclic resolution theorem)**.** *Let $T^\cdot : \mathcal{C}_1 \to \mathcal{C}_2$ be a universal cohomological functor. Given $J \in \mathcal{C}_1$, suppose $0 \to A^0 \to A^1 \to \cdots$ is a complex in $\mathcal{C}_1$ with each $A^\cdot$ acyclic, $h^0(A^\cdot) \cong J$, and $h^i(A^\cdot) = 0$ for $i > 0$. (That is, this complex is an* acyclic resolution *of J.) Then for each $i \geq 0$, there is an isomorphism $T^i(h^0(A^\cdot)) \cong h^i(T^0(A^\cdot))$ which is functorial in the input data.*

5 Derived functors

We are now ready to make some universal cohomological functors. Unfortunately, we are in a bit of a jam: we would like to define them using acyclic resolutions, but the definition of an acyclic object depends on the definition of the cohomological functor. We get out of this vicious circle by identifying some objects which are *always* acyclic.

An object X in an abelian category $\mathcal{C}$ is *injective* if the functor $\mathrm{Hom}(\cdot, X) : \mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Ab}}$ is exact. Since this functor is already left exact, it is enough to require something weaker: if $0 \to Y \to Z$ is a monomorphism, then for any morphism $Y \to X$ we can find some morphism $Z \to X$ fitting into the diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[dr] & Z \arrow[d, dashed] \\
& & X
\end{tikzcd}$$

For instance, in $\underline{\mathrm{Ab}}$, an object X is injective if and only if it is *divisible*, i.e., the multiplication-by-n maps for each positive integer n are all surjective. You might be more familiar with the dual notion: an object X in an abelian category $\mathcal{C}$ is *projective* if the functor $\mathrm{Hom}(X, \cdot) : \mathcal{C} \to \underline{\mathrm{Ab}}$ is exact. In $\underline{\mathrm{Mod}}_R$, any *free* module is projective; in fact, a module is projective if and only if it is a direct summand of a free module.

Lemma. *Any short exact sequence*

$$0 \to I \to B \to C \to 0$$

with I injective is split, *i.e., there exists an arrow $C \to B$ such that $C \to B \to C$ is an isomorphism.*

Proof. Apply the definition of injectivity to the monomorphism $I \to B$ and the arrow $I \to I$ to get a map $B \to I$ such that $I \to B \to I$ is the identity. Then the kernel of $B \to I$ will be isomorphic to C. □

We once again hit a distinction between non-arrow-theoretic and arrow-theoretic conditions; while the property of being a short exact sequence is not preserved under an arbitrary additive functor, the property of being *split* short exact is. That is because a splitting of $0 \to A \to B \to C \to 0$ specifies a pair of endomorphisms $e_1, e_2 : B \to B$ whose sum is B, namely $B \to A \to B$ and $B \to C \to B$, and conversely these endomorphisms determine the sequence.

Proposition. *Let $T^\cdot$ be a cohomological functor such that T^i is effaceable for $i > 0$ (so in particular it is universal). Then for any injective object I, $T^i(I) = 0$ for $i > 0$.*

Proof. Choose a monomorphism $u : I \to B$ with $T^i(u) = 0$, then form the short exact sequence

$$0 \to I \to B \to C \to 0.$$

Since this sequence splits, the resulting sequences

$$0 \to T^j(I) \to T^j(B) \to T^j(C) \to 0$$

are exact for all j. Consequently, the connecting homomorphism $\delta^{i-1} : T^{i-1}(C) \to T^i(I)$ is zero. On the other hand, the morphism $T^j(I) \to T^j(B)$ is just $T^j(u)$, which is also zero. So the exactness of the sequence $T^{i-1}(C) \xrightarrow{\delta^{i-1}} T^i(I) \xrightarrow{T^i(u)} T^i(B)$ forces $T^i(I) = 0$. □

This more or less forces us into the following definition. We say that the category $\mathcal{C}$ *has enough injectives* if for any object X there exists a monomorphism $X \to I$ with I injective. Then any universal cohomological functor can be computed using *injective resolutions*. On the other hand, given an object X, we can always find an injective resolution; better yet, given any morphism $X \to Y$ and an injective resolution of X, we can find an injective resolution of Y and a morphism inducing $X \to Y$ on cohomology. This suggests that we define the *right derived functors* of a left exact functor F by saying for any object X, if $I^\cdot$ is an injective resolution of X, put

$$R^i F(X) = h^i(F(I^\cdot)).$$

Theorem. *Assume that $\mathcal{C}$ has enough injectives. Then the previous definition gives a well-defined cohomological functor, which is effaceable and hence universal.*

The effaceability is obvious from the fact that injectives are acyclic under this definition (if X is injective, use $0 \to X \to 0 \to \cdots$ as the injective resolution). The hard part, or rather the easy but tedious part, is to check that what you are writing down is really a well-defined cohomological functor in the first place. This is so tedious I won't even make you do it as an exercise; rather, I've just asked you to list which compatibilities need to be checked in the first place, which is already a nontrivial effort.

6 Examples

Here are some possibly familiar examples of derived functors. Some of these admit reasonable explicit computations; see exercises.

For $X \in \underline{\mathrm{Mod}}_R$, $X \otimes \cdot$ is a right exact covariant functor from $\underline{\mathrm{Mod}}_R$ to $\underline{\mathrm{Mod}}_R$, hence a left exact covariant functor from $\underline{\mathrm{Mod}}_R^{\mathrm{op}}$ to $\underline{\mathrm{Mod}}_R^{\mathrm{op}}$. The derived functors are called $\mathrm{Tor}^i(X, \cdot)$.

Proposition. *For $X \in \underline{\mathrm{Mod}}_R$, the following are equivalent.*

(a) *X is flat.*

(b) $\mathrm{Tor}^i(X, Y) = 0$ *for any $i > 0$ and any $Y \in \underline{\mathrm{Mod}}_R$.*

(c) $\mathrm{Tor}^1(X, Y) = 0$ *for any $Y \in \underline{\mathrm{Mod}}_R$.*

Proof. Given (a), the functor $X \otimes \cdot$ is exact, so its derived functors are zero, proving (b). Given (b), (c) is trivial. Given (c), for any short exact sequence $0 \to A \to B \to C \to 0$, we get a long exact sequence

$$0 \to X \otimes A \to X \otimes B \to X \otimes C \to \mathrm{Tor}^1(X, A) = 0$$

so $X \otimes A$ is exact, proving (a). $\square$

This is of course a totally general argument: if F is a left exact covariant functor, then F is exact iff $R^i F = 0$ identically for all $i > 0$ iff $R^1 F = 0$ identically.

Given that the tensor product is symmetric, one would like to identify $\mathrm{Tor}^i(X, Y)$ with $\mathrm{Tor}^i(Y, X)$. However, the definition of Tor is asymmetric, so this takes a bit of thinking (which I'll do using the dual language of *projective resolutions* and *homology* and lower indices, but you can switch back if you like). Before starting, note that at least the fact that $\mathrm{Tor}^i(X, Y)$ is functorial in X (not just in Y) is clear from the universal property of universal cohomological functors.

Let $P.$ and $Q.$ be *projective resolutions* of X and Y, respectively. Then we have a double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & P_1 \otimes Q_1 \arrow[r] \arrow[d] & P_1 \otimes Q_0 \arrow[r] \arrow[d] & P_1 \otimes Y \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_0 \otimes Q_1 \arrow[r] \arrow[d] & P_0 \otimes Q_0 \arrow[r] \arrow[d] & P_0 \otimes Y \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & X \otimes Q_1 \arrow[r] \arrow[d] & X \otimes Q_0 \arrow[r] \arrow[d] & X \otimes Y \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the homology of the bottom row computes $\mathrm{Tor}^i(X, Y)$, the homology of the right column computes $\mathrm{Tor}^j(Y, X)$, and the other rows and columns are exact.

It is now a diagram chase to check that we have canonical isomorphisms $\mathrm{Tor}^i(Y, X) \cong \mathrm{Tor}^i(X, Y)$. For instance, say I start with a class in $\mathrm{Tor}^1(X, Y)$ represented by $x \in X \otimes Q_1$. Lift x to $P_0 \otimes Q_1$, then push to $P_0 \otimes Q_0$. The result maps to 0 in $X \otimes Q_0$, so lifts to $P_1 \otimes Q_0$; push to $P_1 \otimes Y$ to get a class in $\mathrm{Tor}^1(Y, X)$. (This is really an example of a *spectral sequence*; more on those a bit later.)

Corollary. *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of R-modules with A_3 flat. Then for any R-module M, $0 \to M \otimes A_1 \to M \otimes A_2 \to M \otimes A_3 \to 0$ is again exact.*

Proof. We have a long exact sequence

$$\mathrm{Tor}^1(M, A_1) \to M \otimes A_1 \to M \otimes A_2 \to M \otimes A_3 \to 0$$

but the left term can be identified with $\mathrm{Tor}^1(A_1, M)$, which vanishes because A_1 is flat. $\square$

The example of Tor is particularly important in algebraic geometry because of *Serre's intersection multiplicity formula.* Let X be a regular excellent scheme, let Y, Z be two integral closed subschemes defined by the ideal sheaves $\mathcal{I}, \mathcal{J}$, and let x be the generic point of a component of $Y \cap Z$. The *naïve intersection multiplicity* of Y and Z at x is

$$\mathcal{O}_{Y \cap Z, x} = \mathcal{O}_{X,x}/(\mathcal{I}\mathcal{J})_x,$$

and this gives the correct answer when $\dim(X) = 2$, $\dim(Y) = \dim(Z) = 1$ (meaning the answer that makes Bézout's theorem work) but not in general. Serre found that the "right" multiplicity is

$$\sum_i (-1)^i \operatorname{length}_{\mathcal{O}_{X,x}} \mathrm{Tor}^i_{\mathcal{O}_{Z,x}}(\mathcal{O}_{X,x}/\mathcal{I}_x, \mathcal{O}_{X,x}/\mathcal{J}_x).$$

It was an open question for a long time to give a "geometric" interpretation of the Tor contributions in this formula; such an interpretation was recently provided by Jacob Lurie using *derived algebraic geometry*. (Roughly speaking, one replaces rings by certain topological rings before applying Spec; the intersection multiplicity then appears as the Euler characteristic of the "derived schematic intersection".)

A similar example occurs using the bifunctor Hom, except that it is really a bifunctor from $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. (Here $\mathcal{C}$ can be *any* abelian category, not just $\underline{\mathrm{Mod}}_R$.) Anyway, the right derived functors of $\mathrm{Hom}(X, \cdot)$ are called $\mathrm{Ext}^i(X, \cdot)$, and they also occur as derived functors of $\mathrm{Hom}(\cdot, Y)$ by the double complex argument (with arrows appropriately reversed).

One more important example: if G is a group (considered with the discrete topology, if you must), let $\mathbb{Z}[G]$ be the *group algebra* of G with coefficients in $\mathbb{Z}$, i.e., additively the direct sum $\oplus_{g \in G} \mathbb{Z}[g]$ with $\mathbb{Z}$-linear multiplication characterized by $[g][h] = [gh]$. Then the covariant functor $\cdot^G : \underline{\mathrm{Mod}}_{\mathbb{Z}[G]} \to \underline{\mathrm{Mod}}_{\mathbb{Z}}$ computing G-invariants is left exact; its derived functors are called *group cohomology* and denoted $H^{\cdot}(G, M)$. The covariant functor $\cdot_G : \underline{\mathrm{Mod}}_{\mathbb{Z}[G]} \to \underline{\mathrm{Mod}}_{\mathbb{Z}}$ computing G-coinvariants (i.e., M maps to the quotient of M by $g(m) - m$ for all $g \in G$ and $m \in M$) is right exact; its derived functors are called *group homology* and denoted $H_{\cdot}(G, M)$. These are actually special cases of the previous example, namely

$$H^{\cdot}(G, M) = \mathrm{Ext}^i_{\underline{\mathrm{Mod}}_{\mathbb{Z}[G]}}(\mathbb{Z}, M), \qquad H_{\cdot}(G, M) = \mathrm{Tor}_i^{\underline{\mathrm{Mod}}_{\mathbb{Z}[G]}}(\mathbb{Z}, M).$$

(More generally, one could replace $\mathbb{Z}$ with an arbitrary ring.)