18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Introduction

In this lecture, I'll give a bit of an overview of what we will be doing this semester, and in particular how it will differ from 18.725. We will start in earnest (with the rudiments of category theory) in the next lecture.

1 Where we were, and where we need to go

In 18.725, we studied the notion of an *abstract algebraic variety* over an algebraically closed field. This combines a lot of the commutative algebra developed in the early 20th century (largely to explain the geometric reasoning of the masters of the Italian school) with Weil's fundamental idea to glue affine algebraic varieties in the same way that one glues local charts together to build manifolds. So what's left?

- We would like to deal with phenomena of *nonreducedness*, for instance as it emerges under *degenerations*. One of the key ideas of the Italian school for understanding things like the geometry of the moduli space of curves was to notice that if you have a family of algebro-geometric objects defined in terms of a parameter t, then the behavior of a particular member of the family is sometimes much simpler than that of a general member. For instance, for a general t, the elliptic curve $y^2 = x^3 + tx + t$ does not have a rational parametrization, but it does in the special case $t = 0$. One can often understand something about the general member of the family by first analyzing a special member, then figuring out how the information you are looking for gets transmitted back to the general member via the degeneration.

  However, degenerations of algebraic varieties are not always best viewed as algebraic varieties. For example, if $t \neq 0$, then the homogeneous polynomial $y^2 - tx^2$ in x, y, z (over, say, the complex numbers) defines a pair of lines. The degeneration at $t = 0$, however, is the *single* line $y = 0$, because the equations $y^2 = 0$ and $y = 0$ define the same variety. In order for the degeneration to preserve the degree of the curve, we need to remember that it is y^2 rather than y which defines this line. That is, the function y on the variety should be *nilpotent*, a possibility that is not afforded by the category of algebraic varieties.

- We would like to work over fields which are not *algebraically closed*. The restriction to algebraically closed fields was originally needed to make things like Bézout's theorem work. However, at the end of the day, we are sometimes interested in solving polynomials over non-algebraically closed fields. For instance, the elliptic curves $y^2 = x^3 + tx$ defined by rational numbers t are all isomorphic as algebraic varieties over the complex numbers. However, they have rather different arithmetic behaviors; for instance, the curve for $t = 1$ has only finitely many rational points, whereas the curve for $t = 73$ has infinitely many.

Weil had an answer for this point: he suggested embedding one's given base field in a large algebraically closed field, called a *universal domain.* However, Weil's answer looks like a mistake in hindsight, because it is not sufficiently *functorial*; see below.

- We would also like to work over (commutative, unital) *rings*, not just fields. For instance, already in Weil's work the question of *reduction mod p* arises, but cannot be addressed while working over fields.

  Even in the context of varieties, one often wants to work over a base which is not a field. For instance, the theory of *elliptic surfaces* is largely thought of by viewing these surfaces as (relative) elliptic curves over a base curve.

There's more, but enough for now.

2 Paradigm shift 1: sheaves

At the time, one might have expected that the future development of algebraic geometry would proceed as a natural descent from Weil's 1946 *Foundations*, with more bells and whistles attached to extend generality. However, just as the theory of epicycles to explain the motion of planets was thrown into disrepute by two paradigm shifts (Galileo's heliocentricity and Kepler's elliptic orbits), two paradigm shifts rendered Weil's foundations a dead end in the development of algebraic geometry. (Most material written in that language has since appeared in modern terminology; what remains untranslated is as intelligible to the modern reader as Chaucer's Middle English.)

The first of these shifts can be attributed to Serre, who introduced the notion of *sheaves* into algebraic geometry. These are the sort of objects defined by descriptions like "take all continuous functions on all open subsets of a topological space", or "take all differentiable functions on all open subsets of a smooth manifold". The latter example is particularly helpful to keep in mind: it is possible to have two different smooth manifolds which are isomorphic as topological spaces (e.g., to $\mathbb{R}^4$, or to a seven-dimensional sphere), but not as smooth manifolds. That is, the underlying topological space does not carry enough information to detect nonisomorphism of smooth manifolds. However, the sheaf of differentiable functions does carry enough information.

Sheaves were originally introduced by Cartan in order to simplify and extend the theory of complex analytic geometry. It is Serre who recognized their place in modern algebraic geometry, by observing (among other things) that they give you a natural way to add nilpotents. In my example of the lines $y = 0$ versus $y^2 = 0$ in the (x, y)-plane, it will turn out that (in the category of schemes) the underlying sets of these two objects will be the same, but the sheaves of regular functions will differ.

However, it will take us some time before we can relate sheaves to algebraic geometry. We will first have to take some time to discuss topological spaces equipped with rings of "interesting" functions, giving rise to the notion of a *locally ringed space.* This notion includes many familiar things: topological spaces, topological manifolds, smooth manifolds, and even abstract algebraic varieties.

But what we really want to include into this category is the prime spectrum of an arbitrary (commutative) ring. Recall that over an algebraically closed field, by the Nullstellensatz there is a bijection between the points of an affine algebraic variety and the *maximal* ideals of its ring of regular functions. For a general ring, Zariski suggested to instead look at the set of *prime* ideals, i.e., the *prime spectrum* of the ring; that way, any map of rings would correspond to a map (contraction) on prime ideals in the opposite direction.

The "fundamental theorem of schemes" is that this set carries the natural structure of a *sheaf* of rings. In other words, the prime spectrum of a ring can be viewed as a locally ringed space. With that (nontrivial) fact in hand, we will be ready to glue prime spectra together to manufacture arbitrary schemes.

3 Paradigm shift 2: functors

The second paradigm shift that stood between Weil and modern algebraic geometry is mostly due to Grothendieck, though it is of a piece with the formalist view of mathematics propounded by the Bourbaki school of French mathematicians in the middle of the 20th century. It is to conceive of algebraic geometry in the language of *categories* and *functors*. Roughly speaking, a *category* is the collection of all mathematical objects of a given type, equipped with the maps between those objects that preserve the distinguishing structures. The key example to keep in mind is the category of all rings, together with all homomorphisms between rings.

At first, it may seem rather a bad idea to deal with categories; for one thing, they cannot be viewed as sets due to some annoying paradoxes in set theory (such as Russell's paradox). But once you get past such considerations, dealing with categories is not so hard, and in fact they appear everywhere around you.

Here is where categories appear naturally in algebraic geometry. Say $P_1, \dots, P_m$ are polynomials in the variables $x_1, \dots, x_n$ over a ring R. Then for *any* ring S equipped with a homomorphism $R \to S$, it makes sense to consider the set

$$\{(x_1, \dots, x_n) \in S^n : P_1(x_1, \dots, x_n) = \cdots = P_m(x_1, \dots, x_n) = 0\}$$

of S-valued solutions to the system of equations $P_1 = \cdots = P_m = 0$. One should thus avoid linking these polynomials to a single set of "points", but instead view them as a *scheme* for converting rings into sets of solutions. This gives a natural example of a *functor* between two categories, i.e., a rule for converting objects of one category into objects of the other, and for converting morphisms between two objects of the first category into morphisms between the image objects of the second category. (In our example, we are converting R-algebras into sets.)

One benefit of this point of view is that it naturally distinguishes, for instance, the zero loci of y and y^2: they give the same sets when we plug in an algebraically closed field k, but not when we plug in a ring such as $k[\epsilon]/(\epsilon^2)$.

That benefit by itself is not so significant, as it still doesn't really prove that category theory is good for anything other than formulating simple statements in complicated language. What makes category theory so useful, and how we will exploit it in our work, is that it lets you formalize certain types of "reasoning by analogy" that mathematicians would like to engage in all the time, but which is sometimes difficult. One key example in the context of schemes is the notion of a *product*. Given two mathematical objects X and Y, how should one define their product $X \times Y$? When X and Y are given as sets carrying some extra structure (e.g., groups, rings, etc.), the correct answer is to take the Cartesian product of the underlying sets and then somehow cook up a good structure on that.

From the point of view of category theory, though, the right way to answer this question is to specify a *universal property* that should be satisfied by the product. Namely, the product $X \times Y$ should have the following properties.

(a) It should come with projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$.

(b) Given any object Z, the function taking a map $f : Z \to X \times Y$ to the pair of compositions $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ g : Z \to Y$ should be a bijection.

This does not by itself actually construct products; indeed, some categories may not always admit product objects according to this definition. However, it does give a characterization of how a "correct" definition of a product object should behave. In fact, it's okay to come up with two different definitions as long as they both satisfy the universal property; the effect is that there will be *canonical identifications* between the two types of projects.

We will use this particular example to construct products in the category of schemes. There, we will discover that the product of two schemes does *not* have underlying set equal to the Cartesian products of the underlying sets!