18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
More properties of schemes (updated 9 Mar 09)

I've now spent a fair bit of time discussing properties of morphisms of schemes. However, there are a few properties of individual schemes themselves that merit some discussion (especially for those of you interested in arithmetic applications); here are some of them.

1 Reduced schemes

I already mentioned the notion of a *reduced* scheme. An affine scheme $X = \mathrm{Spec}(A)$ is *reduced* if A is a reduced ring (i.e., A has no nonzero nilpotent elements). This occurs if and only if each stalk $A_{\mathfrak{p}}$ is reduced. We say X is *reduced* if it is covered by reduced affine schemes.

Lemma. *Let X be a scheme. The following are equivalent.*

(a) X is reduced.

(b) For every open affine subsheme $U = \mathrm{Spec}(R)$ of X, R is reduced.

(c) For each $x \in X$, $\mathcal{O}_{X,x}$ is reduced.

Proof. A previous exercise. $\square$

Recall that any closed subset Z of a scheme X supports a unique *reduced* closed subscheme, defined by the ideal sheaf $\mathcal{I}$ which on an open affine $U = \mathrm{Spec}(A)$ is defined by the intersection of the prime ideals $\mathfrak{p} \in Z \cap U$. See Hartshorne, Example 3.2.6.

2 Connected schemes

A nonempty scheme is *connected* if its underlying topological space is connected, i.e., cannot be written as a disjoint union of two open sets. (The empty scheme is not connected.)

Lemma. *The scheme X is connected if and only if the idempotent elements of $\Gamma(X, \mathcal{O}_X)$ (i.e., the solutions of $e = e^2$) are 0 and 1.*

Proof. If X is a disjoint union of open sets U and V, then we can construct an idempotent $e \neq 0, 1$ by taking the pullback of 0 along $U \to \mathrm{Spec}\,\mathbb{Z}$ and the pullback of 1 along $V \to \mathrm{Spec}\,\mathbb{Z}$. Conversely, if $e \in \Gamma(X, \mathcal{O}_X)$ is an idempotent, then its value at each $x \in X$ is either 0 or 1; the sets where the two values occur are closed and form a partition of X, so X is disconnected. $\square$

In many reasonable cases, X can be written as a disjoint union of connected open subschemes; these are then called the *connected components* of X.

3 Irreducible schemes

A nonempty scheme is *irreducible* if its underlying topological space is irreducible, i.e., cannot be written as a union of two proper closed subsets, i.e., does not contain two disjoint nonempty open subsets. (The empty scheme is not irreducible.) Note that a nonempty open subscheme of an irreducible scheme is still irreducible.

Lemma. *The nonempty affine scheme $X = \mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal (i.e., every zero divisor of A is nilpotent).*

Proof. Note that X is irreducible if and only if the intersection of any two nonempty open subsets is nonempty. It is of course enough to check the intersection of two distinguished opens $D(f), D(g)$. They are nonempty if and only if f and g are not nilpotent; the intersection $D(fg)$ is nonempty if and only if fg is not nilpotent. Hence X is irreducible if and only if the nilradical of A is prime. $\square$

Handy fact: the spectrum of a local ring is irreducible, because the maximal ideal belongs to every closed subset.

A *generic point* of a topological space is a point belonging to every nonempty open subset.

Lemma. *If X is irreducible, then X has a unique generic point.*

Proof. If $X = \mathrm{Spec}(A)$, then the nilradical of A is the unique generic point. In general, if X is irreducible and $U = \mathrm{Spec}(A)$ is a nonempty open affine, then any generic point of X is also a generic point of U. Conversely, if $\eta \in U$ is the unique generic point of U (which exists because U is forced to be irreducible), then there cannot be an open affine subset V of X omitting η, as then $V \cap U$ would have to be empty (since it is an open subset U missing the generic point of U), a contradiction. $\square$

4 Integral schemes

A nonempty scheme is *integral* if it is irreducible and reduced. (The empty scheme is not irreducible.)

Lemma. *Put $X = \mathrm{Spec}(A)$. Then the following are equivalent.*

(a) X is integral.

(b) A is an integral domain. (The zero ring is not an integral domain.)

(c) X is connected and each local ring $\mathcal{O}_{X,x}$ is an integral domain.

Proof. The only nontrivial implication is (c) $\Longrightarrow$ (a). Suppose (c); note that it implies that X is reduced. Choose $f \in A$. Let U be the set of $x \in X$ such that f has nonzero image in $\mathcal{O}_{X,x}$; then U is open (previously assigned exercise).

We claim that $X \setminus U$ is also open. To see this, pick $x \in X \setminus U$ corresponding to a prime ideal $\mathfrak{p}$ of A. Since f maps to zero in $A_\mathfrak{p}$, there must exist $g \in A \setminus \mathfrak{p}$ for which $fg = 0$. That equality in turn implies that $D(g)$, which contains $\mathfrak{p}$, is in fact contained in $X \setminus U$. Since each point of $X \setminus U$ has an open neighborhood contained in $X \setminus U$, we conclude that $X \setminus U$ is open.

Since X is connected, it follows that U equals either X or the empty set. In the latter case, f belongs to the nilradical of A, and so equal 0 because X is reduced.

We conclude that if $f, g \in A$ are nonzero, their images in each $A_\mathfrak{p}$ are nonzero. Hence fg also has nonzero image in each $A_\mathfrak{p}$, and so must be nonzero. This proves (a). $\square$

5 Normal schemes

A scheme X is *normal* if for each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integral domain and is integrally closed in its field of fractions.

Lemma. *Suppose $X = \mathrm{Spec}(A)$ is connected. Then X is normal if and only if A is an integral domain which is integrally closed in its field of fractions.*

Proof. If A is an integral domain which is integrally closed in its field of fractions, then so is each localization of A (see Atiyah-Macdonald, Proposition 5.12), so X is normal. Conversely, suppose X is connected and normal. By the previous lemma, A is an integral domain.

It remains to check that an integral domain is integrally closed (in its field of fractions) if and only if its localization at each prime ideal has this property. This follows from the easy fact that A is the intersection of the $A_\mathfrak{p}$. $\square$

The construction of the integral closure of a domain can be sheafified. (Note: a *dominant* morphism is one with dense image.)

Theorem 1. *Let X be an integral scheme. Then the category of dominant morphisms $\tilde{X} \to X$ with $\tilde{X}$ normal has a final element.*

Proof. Exercise. $\square$

The final element is called the *normalization* of X. Under "normal" circumstances, the morphism $\tilde{X} \to X$ is finite, but there are pathological counterexamples unless one imposes some hypotheses.

One attempt is the notion of a *Nagata ring*. We say an integral domain R is *N-1* if the integral closure of R in $\mathrm{Frac}(R)$ is finite as an R-module. We say R is *N-2* if for any finite extension L of $\mathrm{Frac}(R)$, the integral closure of R in L is finite as an R-module. We say a general ring R is a *Nagata ring* if R is noetherian and $R/\mathfrak{p}$ is N-2 for any prime ideal $\mathfrak{p}$ of R. (Without the noetherian hypothesis, I think this is what is called a *universally Japanese* ring in EGA. My definition is from Matsumura, *Commutative Algebra*, §31.) The point is that the Nagata property is stable under many natural operations: localizations, quotients, passing to a finitely generated ring extension, certain types of completion, etc.

6 Dimension and codimension

The *dimension* of a scheme X is the length of the longest chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X (keeping in mind that the numbering starts at 0). The dimension of an affine scheme $X = \operatorname{Spec}(A)$ is the same as the *Krull dimension*, since irreducible closed sets of X correspond to prime ideals of A.

The *codimension* of an irreducible closed subset Z of X is the length of the longest chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X for which $Z_0 = Z$. We can similarly define the codimension of one irreducible closed subset inside another.

These notions can behave badly even for the spectrum of a noetherian ring (Hartshorne, Caution 3.2.8). Again, we need to impose more hypotheses before working with these in any detail; the best way to do this is work with the class of *excellent schemes*. More on those later.

7 Regular schemes

Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. The *cotangent space* of A is the k-vector space $\mathfrak{m}/\mathfrak{m}^2$; its dual is called the *tangent space* of A.

Suppose A is noetherian. Then it is a nontrivial theorem from commutative algebra (e.g., Matsumura §12) that

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim A.$$

If equality holds, we say that A is *regular*.

We say that a scheme X is *regular* at a point x if $\mathcal{O}_{X,x}$ is a regular local ring, and simply *regular* if it is regular everywhere. For instance, if X is a scheme of finite type over a field k, then X is regular if and only if the corresponding variety is *nonsingular* everywhere. For another example, $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{Z}[x]$ are both regular. We will give a relative version of nonsingularity later (the notion of a *smooth morphism*).

8 Excellent rings and schemes

A *quasiexcellent ring* is a noetherian ring R with the following properties.

(a) For any prime ideal $\mathfrak{p}$ of R and any homomorphism $R \to K$ with K a field, the ring $\hat{R}_{\mathfrak{p}} \otimes_A K$ is regular.

(b) Any integral domain A which is finite as an R-algebra is *generically regular*, i.e., there exists $a \in A$ nonzero such that A_a is regular.

An *excellent ring* is a quasiexcellent ring R with the following additional property.

(c) The ring R is *universally catenary*. That is, for any nonnegative integer n and any two prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ of $R[x_1, \ldots, x_n]$, any two maximal chains of prime ideals of $R[x_1, \ldots, x_n]$ starting with $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have the same length.

The class of excellent rings is introduced by Grothendieck in EGA IV part 3 (see §7.8). It includes some natural examples (fields, $\mathbb{Z}$, complete local rings, and the series in $\mathbb{C}[\![x_1, \ldots, x_n]\!]$ convergent in a neighborhood of the origin) and is stable under nice operations (localization, completion, quotient, polynomial ring). These rings have lots of useful properties: for instance, they are Nagata rings.