

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Morphisms of schemes (updated 20 Feb 09)

We next introduce morphisms of locally ringed spaces and schemes. Same references as the previous handout.

Missing remark from last time: when EGA was written, what we now call a scheme was called a *prescheme*. EGA's "scheme" is what we will call a *separated scheme*.

1 Direct and inverse image

To define morphisms of ringed spaces, I need the direct and inverse image functors for sheaves. I had these on a previous handout, but didn't discuss them in class, so let me review.

Let $f : X \rightarrow Y$ be a continuous map. For \mathcal{F} a sheaf on X , the formula

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

obviously defines a sheaf $f_*\mathcal{F}$ on Y . It is called the *direct image* of \mathcal{F} .

Now let \mathcal{G} be a sheaf on Y . Define a presheaf $f^{-1}\mathcal{G}$ on X as follows: for U open in X , let $(f^{-1}\mathcal{G})(U)$ be the stalk of \mathcal{G} at $f(U)$, i.e., the direct limit of $\mathcal{G}(V)$ over open sets $V \subseteq X$ containing $f(U)$. This is generally not a sheaf; its sheafification is called the *inverse image* of \mathcal{G} , denoted $f^{-1}\mathcal{G}$. (The notation f^* is reserved for something else; see below.)

Proposition. *The functors f^{-1} and f_* form an adjoint pair.*

Proof. Exercise. □

Using the inverse image, we can define the *restriction* of \mathcal{F} to an arbitrary subset Z of X , as the sheaf $i^{-1}\mathcal{F}$ for $i : Z \rightarrow X$ the inclusion map (with Z given the subspace topology). If $Z = \{x\}$, this coincides with the stalk \mathcal{F}_x (exercise).

2 Morphisms of (locally) ringed spaces

Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be ringed spaces. A *morphism* of ringed spaces from X to Y consists of a continuous map $f : X \rightarrow Y$ plus a homomorphism $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ of sheaves of rings on Y .

For example, if X and Y are manifolds, then $f^\#$ acts as follows. Given an open subset U of Y , we must specify a homomorphism from $\mathcal{O}_Y(U)$ to $f_*(\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$. This homomorphism is pullback by f ; that is, it takes a continuous function $g : U \rightarrow \mathbb{R}$ to the continuous function $g \circ f : f^{-1}(U) \rightarrow \mathbb{R}$.

In this example, both X and Y are locally ringed spaces. Moreover, the homomorphism $f^\#$ has an important property not implied by the definition of a morphism of ringed spaces: if $g : U \rightarrow \mathbb{R}$ vanishes at some point $y \in Y$, then $g \circ f$ vanishes at any point $x \in X$ for which $f(x) = y$. More generally, the value of g at y equals the value of $g \circ f$ at x .

This is such an important property that we build it into the definition of a morphism of locally ringed spaces. If (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are locally ringed spaces, a *morphism* of locally ringed spaces is a morphism of the underlying ringed spaces such that for each point $x \in X$ mapping to $y \in Y$, the induced homomorphism $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ of local rings is a *local homomorphism*, that is, the inverse image of $\mathfrak{m}_{X,x}$ is $\mathfrak{m}_{Y,y}$.

3 Morphisms of schemes

A *morphism* between two schemes is simply a morphism between the underlying locally ringed spaces. E.g., for $U \subseteq X$ open, restricting X to U gives a locally ringed space which is again a scheme, and the inclusion is a morphism of schemes. Such a U is called an *open subscheme* of X .

It may be a surprise that merely requiring morphisms of schemes to preserve the locally ringed space structure has the expected effect.

Theorem 1. *For A and B two rings, the set of morphisms $(f, f^\#) : \text{Spec}(A) \rightarrow \text{Spec}(B)$ of locally ringed spaces corresponds bijectively to the set of ring homomorphisms $f^* : B \rightarrow A$, where $(f, f^\#)$ goes to the map*

$$f^\#(\text{Spec}(A)) : \Gamma(\text{Spec}(B), \mathcal{O}_{\text{Spec}(B)}) = B \rightarrow \Gamma(\text{Spec}(B), f_*(\mathcal{O}_{\text{Spec}(A)})) = A.$$

In fact, something more general is true.

Theorem 2. *Let $\underline{\text{LocRingSp}}$ be the category of locally ringed spaces. For any locally ringed space (X, \mathcal{O}_X) and any ring A , there is a natural bijection*

$$\text{Hom}_{\underline{\text{LocRingSp}}}((X, \mathcal{O}_X), \text{Spec}(A)) \rightarrow \text{Hom}_{\underline{\text{Ring}}}(A, \Gamma(X, \mathcal{O}_X))$$

obtained by taking global sections. In other words, the functors Spec and $\Gamma(\cdot, \mathcal{O}_\cdot)$ from the category of rings to the opposite category of locally ringed spaces form an adjoint pair.

Proof. We first define the inverse map. Given a map $f^* : A \rightarrow \Gamma(X, \mathcal{O}_X)$ and a point $x \in X$, let $f(x)$ be the point of $\text{Spec}(A)$ corresponding to the inverse image of $\mathfrak{m}_{X,x}$ under the composition $A \rightarrow \Gamma(X, \mathcal{O}_X) \rightarrow \mathcal{O}_{X,x}$.

To see that f is continuous, it is enough to check that the inverse image of a distinguished open subset $D(g)$ is open. But this inverse image consists of the points $x \in X$ where $f^*(g) \in \Gamma(X, \mathcal{O}_X)$ has a nonzero value, and this is indeed open. Better yet, if $x \in X$ is a point where $f^*(g)$ has a nonzero value, then (since $\mathcal{O}_{X,x}$ is a local ring) g has a multiplicative inverse in $\mathcal{O}_{X,x}$, and so has a multiplicative inverse everywhere in some open neighborhood of x . As a corollary, we observe that g has a multiplicative inverse everywhere on $f^{-1}(D(g))$ (since the local inverses are unique, and hence must glue).

Now that f is known to be continuous, we can define $f^\#$. It is sufficient to define it on distinguished opens, that is, we must specify

$$f^\#(D(g)) : \Gamma(D(g), \mathcal{O}_{\text{Spec}(A)}) = A_g \rightarrow \Gamma(D(g), f_*(\mathcal{O}_X)) = \Gamma(f^{-1}(D(g)), \mathcal{O}_X).$$

To do this, write any $h \in A_g$ as a/g^i with $a \in A$ and $i \in \mathbb{Z}_{\geq 0}$. We can then map a to $\Gamma(X, \mathcal{O}_X)$ and then by restriction to $\Gamma(f^{-1}(D(g)), \mathcal{O}_X)$. By the previous paragraph, g maps to a unit in $\Gamma(f^{-1}(D(g)), \mathcal{O}_X)$, so we can compute (unambiguously) the image of a/g^i .

It is clear that if we start with a ring homomorphism, then pass to locally ringed spaces, then return, we get back the original ring homomorphism. The hard part is to check that if we start with a morphism of locally ringed spaces on the left, then go to the right and come back, we get back the morphism we started with. What we need the extra condition for is to see that the underlying map on topological spaces is reproduced; once that holds, we get the equality of homomorphisms of ring sheaves by comparing them on stalks. \square

Here is a simple example to illustrate why we need morphisms of locally ringed spaces, rather than ringed spaces. Let R be a discrete valuation ring with fraction field K , e.g., $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$. Then $\text{Spec}(K)$ consists of a single point (0) , while $\text{Spec}(R)$ consists of two points (0) and \mathfrak{m}_R (the maximal ideal) with the first being closed and the second not. The inclusion $R \rightarrow K$ of rings corresponds to a map of locally ringed spaces sending the unique point of $\text{Spec}(K)$ to the point (0) . However, there is also a map of ringed spaces sending $\text{Spec}(K)$ to the point \mathfrak{m}_R and again using $R \rightarrow K$ to define the map on structure sheaves. This is not a morphism of locally ringed spaces because the map $R \rightarrow K$ on stalks is not a local homomorphism. (For the good morphism, the map on stalks is just the identity map $K \rightarrow K$.)

4 Some strange morphisms to schemes

Given any locally ringed space (X, \mathcal{O}_X) , we can use the previous theorem to construct a canonical morphism

$$(X, \mathcal{O}_X) \rightarrow \text{Spec}(\Gamma(X, \mathcal{O}_X))$$

(this is basically an adjunction morphism). This in itself may not be so useful, because X may have very few global functions (e.g., the Riemann sphere with the sheaf of holomorphic functions). On the other hand, if X contains enough global functions to separate points (i.e., if for any $x, y \in X$ we can find $f \in \Gamma(X, \mathcal{O}_X)$ with $f \in \mathfrak{m}_{X,x}$ but $f \notin \mathfrak{m}_{X,y}$), then the canonical homomorphism is injective.

For instance, if X is an affine algebraic variety, this gives a map from X to a scheme. It turns out that this map is a bijection from X to the closed point of the resulting scheme, and in fact gives an embedding of the category of varieties into the category of schemes; see Hartshorne Proposition II.2.6 and the related exercise.

Another example occurs when X is a sufficiently small complex analytic manifold (e.g., a *Stein space*). Such examples will occur when we talk about *analytification* of complex algebraic varieties and Serre's *GAGA principle*.

One other funny but useful example: for X any scheme and $x \in X$, we can construct a morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ by taking any open affine neighborhood U of x in X and performing adjunction on the ring map $\Gamma(U, \mathcal{O}_X) \rightarrow \mathcal{O}_{X,x}$. The result does not depend on

U ; it carries the closed point of $\mathcal{O}_{X,x}$ (the point corresponding to the maximal ideal of the local ring) to x .

5 Fibre products

Recall that a *fibre product* of the morphisms $Y \rightarrow X$ and $Z \rightarrow X$ in any category is a limit of the diagram

$$\begin{array}{ccc} Y & & Z \\ & \searrow & \swarrow \\ & X & \end{array}$$

i.e., a final object among objects mapping to Y and Z making the diagram commute.

We'll construct fibre products of schemes in a moment. First, we observe how fibre products interact with passage to open subschemes.

Lemma. *Suppose $f : Y \rightarrow X$ and $g : Z \rightarrow X$ are morphisms of schemes such that the fibre product $Y \times_X Z$ exists. Let $\pi_1 : Y \times_X Z \rightarrow Y$, $\pi_2 : Y \times_X Z \rightarrow Z$ be the induced maps. Let $T \subseteq X, U \subseteq Y, V \subseteq Z$ be open subsets such that $f(U), g(V) \subseteq T$, viewed as subschemes. Then*

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V),$$

viewed as a subscheme of $Y \times_X Z$, is a fibre product of $U \rightarrow T$ and $V \rightarrow T$. (In particular, the construction does not depend on T .)

Proof. Suppose $S \rightarrow U$ and $S \rightarrow V$ are morphisms such that $S \rightarrow U \rightarrow T$ and $S \rightarrow V \rightarrow T$ agree. Then $S \rightarrow U \hookrightarrow Y \xrightarrow{f} X$ and $S \rightarrow V \hookrightarrow Z \xrightarrow{g} X$ agree, so S factors uniquely through $Y \times_X Z$. Now writing $S \rightarrow U \rightarrow T \hookrightarrow X$ as $S \rightarrow Y \times_X Z \xrightarrow{\pi_1} Y \xrightarrow{f} X$ shows that the image of S in $Y \times_X Z$ lands in $\pi_1^{-1}(U)$; similarly, it lands in $\pi_2^{-1}(V)$. So we get a map $S \rightarrow \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$; conversely, any such map can be composed with the inclusion $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \hookrightarrow X$, so the above argument shows that the map is unique. \square

With this, it is easy to check the existence of fibre products.

Theorem 3. *All fibre products exist in the category of schemes.*

Proof. The easy part is when $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, $Z = \text{Spec}(C)$ are all affine. In that case, the tensor product $B \otimes_A C$ is a fibre coproduct in the category of rings, using the maps $\cdot \otimes 1 : B \rightarrow B \otimes_A C$ and $1 \otimes \cdot : C \rightarrow B \otimes_A C$.

To get the general case, we apply the previous lemma twice. First, if X is affine, then we can cover Y and Z with open affines and use the previous lemma to glue the fibre products. Second, once we know fibre products exist when X is affine, we can cover X itself with open affines (and cover Y and Z with the inverse images of these) and use the lemma again. \square

As noted earlier, this notion of product behaves a bit strangely on the level of sets. For instance, $\text{Spec } \mathbb{R}$ and $\text{Spec } \mathbb{C}$ both contain a single point, but $\text{Spec } \mathbb{C} \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C}$ consists of two points!

6 The functor of points

The previous example illustrates that the set of points of a scheme does not really reflect our geometric intuition, derived largely from our experience with varieties, about the behavior of “points” on geometric objects. A good conceptual workaround for this is the *functor of points*.

Given two schemes S and X , the *set of S -valued points of X* , denoted $X(S)$, is simply the set $\mathrm{Hom}(S, X)$ of morphisms of schemes. If $S = \mathrm{Spec} R$, we may write $X(R)$ instead of $X(S)$. For instance, for any ring R , define the *affine space* $\mathbb{A}_R^n = \mathrm{Spec} R[x_1, \dots, x_n]$. Then for any ring R ,

$$\mathbb{A}_{\mathbb{Z}}^n(R) = \mathbb{A}_R^n(R) = R^n.$$

A more telling example is the fibre product. If $Y \rightarrow X$ and $Z \rightarrow X$ are morphisms, then

$$\mathrm{Hom}(S, Y \times_X Z) = \mathrm{Hom}(S, Y) \times_{\mathrm{Hom}(S, X)} \mathrm{Hom}(S, Z),$$

where the right side denotes the usual fibre product in the category of sets, i.e., you take pairs of morphisms from S , one to Y and one to Z , which agree when you pass to morphisms from S to X .

If we fix X , we may view $X(\cdot)$ as a functor on the category of schemes. There is an appropriate sense in which it is a *sheaf* on that category, but never mind that for now. The one thing you might want to take away from this is that if X is covered by open affines U_i , a morphism $S \rightarrow X$ may not land in any one of the U_i . For instance, $\mathbb{P}_{\mathbb{Z}}^n(S)$ is *not* just obtained by taking the union of the R -valued points of the distinguished open subsets. For instance, the identity morphism $\mathbb{P}_{\mathbb{Z}}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^n$ doesn't occur this way. You can even have this problem for ring-valued points: there is a natural map $\mathrm{Spec} \mathbb{Z}[x_0, \dots, x_n] \rightarrow \mathbb{P}_{\mathbb{Z}}^n$ which does not factor through a distinguished open. (See exercises.)

The functor of points doesn't by itself prove much of anything; for instance, it doesn't tell you how to construct the fibre product. However, it can be used to *suggest* certain natural definitions, e.g., the definition of a *group scheme*. See exercises.

7 Zen and the art of base change

Although the fibre product is a symmetric construction in the two factors, in algebraic geometry we will often use it in an asymmetric fashion. Namely, for $f : Y \rightarrow X$ a morphism and $g : Z \rightarrow X$ another morphism, we refer to $f \times g : Y \times_X Z \rightarrow Z$ as the *base change of f by g* . Geometrically, if you imagine f as giving a family of geometric objects parametrized by X , $f \times g$ describes the pullback of this family to Z .

When we start defining properties of morphisms (next lecture), we will be particularly concerned with their behavior under base change. Typical questions:

- (a) Is the property stable under arbitrary base change? If not, how about base changes where the base change morphism is itself restricted in some way?

- (b) Does the property descend down a suitable base change? E.g., if g is surjective, does $f \times g$ having the property imply the same for f ?

In particular, if (b) is true whenever $g : \coprod_i U_i \rightarrow X$ is a covering of X by open subschemes, we say that the property is *local on the target*. For instance, by our lemma about base change, the property of being injective/surjective on points is local on the target.

We'll give many examples of properties of morphisms later. Here are two to use as a mental model. For $f : Y \rightarrow X$ a morphism of schemes with $X = \operatorname{Spec}(A)$ affine, we say that f is *affine* if $Y = \operatorname{Spec}(B)$ is also affine. I claim that this property satisfies the following condition.

- (i) Let $f : Y \rightarrow X$ be a morphism with X affine. Let $D(g_1), \dots, D(g_n)$ be a finite covering of X by distinguished open subsets. Then f is affine if and only if the induced morphisms $Y \times_X D(g_i) \rightarrow D(g_i)$ are all affine.

This follows from the assigned Hartshorne exercise II.2.17, since the g_i generate the unit ideal in $\Gamma(Y, \mathcal{O}_Y)$. (We'll prove something much stronger in the next lecture.)

It can be deduced from this (exercise) that for $f : Y \rightarrow X$ an arbitrary morphism, the following are equivalent.

- (a) For a *single* open affine cover $\{U_i\}_{i \in I}$ of X , each induced morphism $Y \times_X U_i \rightarrow U_i$ is affine.
- (b) For *each* open affine cover $\{U_i\}_{i \in I}$ of X , each induced morphism $Y \times_X U_i \rightarrow U_i$ is affine. (In other words, for every open affine $U \subseteq X$, the induced morphism $Y \times_X U \rightarrow U$ is affine.)

If these hold, we say that f itself is affine.

In this case, we have an extra condition that is evidently satisfied.

- (ii) Let $f : Y \rightarrow X$ be a morphism with X affine, which is affine. Then for any morphism $g : Z \rightarrow X$ with Z also affine, $f \times g : Y \times_X Z \rightarrow Z$ is also affine.

In this case, (a) and (b) are equivalent to this condition (exercise again).

- (c) For every morphism $g : Z \rightarrow X$ with Z affine, $f \times g : Y \times_X Z \rightarrow Z$ is finite.

Moreover, these equivalent conditions are stable under *arbitrary* base change.

Here is another important example. For $f : Y \rightarrow X$ a morphism of schemes with $X = \operatorname{Spec}(A)$ affine, we say that f is *finite* if $Y = \operatorname{Spec}(B)$ is also affine and B is finite as an A -module. I claim this satisfies (i) and (ii). For (i), we already know that $Y = \operatorname{Spec}(B)$ is affine. Suppose $B \otimes_A A_{g_i}$ is generated as an A_{g_i} -module by some finite set of elements. Since each element is an element of B divided by a power of g_i , we can generate $B \otimes_A A_{g_i}$ as an A_{g_i} -module with a finite subset of B itself. These subsets together generate a finite A -submodule B' such that $(B/B') \otimes_A A_{g_i} = 0$ for each i . That is, the sheaf corresponding to the A -module B/B' is zero; but by a theorem from last time, this forces $B/B' = 0$.

For (ii), note that if $Z = \operatorname{Spec}(C)$, then $Y \times_X Z = \operatorname{Spec}(B \otimes_A C)$ and $B \otimes_A C$ is finite as a C -module: use a set of generators of B as an A -module.

8 Back to schemes for a moment

The strategy I just introduced can be used to establish properties of schemes, not just morphisms, using a trick: to define a property P of schemes, say that a morphism $f : Y \rightarrow X$ has property P if and only if f is an isomorphism and X has property P .

For instance, an affine scheme is *reduced* if its corresponding ring A has no nilpotent elements. This holds if each local ring of A is reduced (exercise), so in particular (i) holds. This allows us to extend the definition of reducedness to arbitrary schemes, and the resulting condition holds if and only if each local ring of the scheme is reduced.

Another approach is to recall that each scheme admits a unique morphism to $\mathrm{Spec}(\mathbb{Z})$, and extract properties of schemes from properties of this morphism. Trivial example: X is affine if and only if $X \rightarrow \mathrm{Spec}(\mathbb{Z})$ is affine.