

**18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)**  
**Serre duality for projective space**

## 1 Ext groups

For  $R$  a ring and  $M, N \in \underline{\text{Mod}}_R$ , I defined  $\text{Ext}^i(M, N)$  as the image of  $N$  under the  $i$ -th right derived functor of  $\text{Hom}_R(M, \cdot)$ . This makes sense because  $\text{Hom}_R(M, \cdot)$  is a left exact covariant functor from  $\underline{\text{Mod}}_R$  to  $\underline{\text{Ab}}$  (it actually goes to  $\underline{\text{Mod}}_R$  but never mind that). I also remarked that it can be viewed as the image of  $M$  under the  $i$ -th right derived functor of  $\text{Hom}_R(\cdot, N)$ , provided we view this as a functor on  $\underline{\text{Mod}}_R^{\text{op}}$ .

For the category  $\underline{\text{Mod}}_X$  of sheaves of  $\mathcal{O}_X$ -modules on a ringed space  $X$ , we can imitate the first construction pretty directly, except that we have to choose between the normal  $\text{Hom}$  and the sheaf  $\text{Hom}$ . Let  $\text{Ext}^i(\mathcal{F}, \cdot)$  be the right derived functors of  $\text{Hom}(\mathcal{F}, \cdot)$ , and let  $\mathcal{E}xt^i(\mathcal{F}, \cdot)$  be the right derived functors of  $\mathcal{H}om(\mathcal{F}, \cdot)$ .

For example, there is a natural isomorphism

$$\text{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F})$$

because these are derived functors of the naturally isomorphic functors  $\text{Hom}(\mathcal{O}_X, \mathcal{F}) \cong H^0(X, \mathcal{F})$ . On the other hand,  $\mathcal{H}om(\mathcal{O}_X, \mathcal{F})$  is the identity functor, so

$$\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}, \quad \mathcal{E}xt^i(\mathcal{O}_X, \mathcal{F}) = 0 \quad (i > 0).$$

**Lemma.** *Let  $\mathcal{I}$  be an injective  $\mathcal{O}_X$ -module. Then for any open subset  $U$  of  $X$ ,  $\mathcal{I}|_U$  is an injective  $\mathcal{O}_U$ -module.*

*Proof.* Let  $j : U \rightarrow X$  be the inclusion. We must show that given a monomorphism  $\mathcal{F} \rightarrow \mathcal{G}$ , any map  $\mathcal{F} \rightarrow \mathcal{I}|_U$  extends to  $\mathcal{G}$ . Let  $j_*$  denote extension by zero, so that  $j_*\mathcal{F}$  has the same stalks as  $\mathcal{F}$  over  $U$  and zero stalks elsewhere. (Sections are the same as  $\mathcal{F}$  over opens contained in  $U$  and zero elsewhere.) By looking at stalks,  $j_*\mathcal{F} \rightarrow j_*\mathcal{G}$  is still a monomorphism. Moreover, we have a map  $j_*\mathcal{I}|_U \rightarrow \mathcal{I}$  by adjunction, and the resulting composition  $j_*\mathcal{F} \rightarrow j_*\mathcal{I}|_U \rightarrow \mathcal{I}$  extends to  $j_*\mathcal{G} \rightarrow \mathcal{I}$ . Restricting back to  $U$  gives the desired map  $\mathcal{G} \rightarrow \mathcal{I}|_U$ .  $\square$

**Corollary.** *For any open subset  $U$  of  $X$ , there are natural isomorphisms*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i(\mathcal{F}|_U, \mathcal{G}|_U).$$

*In particular, the right side is a sheaf; e.g.,  $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = 0$  for  $i > 0$  whenever  $\mathcal{F}$  is locally free of finite rank.*

*Proof.* Both sides are cohomological functors in  $\mathcal{G}$  whose higher terms vanish on injectives (by the previous lemma in the case of the right side), hence are effaceable and thus universal.  $\square$

**Corollary.** For  $\mathcal{I}$  an injective  $\mathcal{O}_X$ -module, the functors

$$\mathrm{Hom}(\cdot, \mathcal{I}), \quad \mathcal{H}om(\cdot, \mathcal{I})$$

are exact.

*Proof.* This is true for  $\mathrm{Hom}$  by the definition of injectivity. For  $\mathcal{H}om$ , use the lemma.  $\square$

**Proposition.** For  $\mathcal{F}$  an  $\mathcal{O}_X$ -module,  $\mathrm{Ext}^i(\cdot, \mathcal{F})$  and  $\mathcal{E}xt^i(\cdot, \mathcal{F})$  are cohomological functors on  $\underline{\mathrm{Mod}}_X^{\mathrm{op}}$ .

*Proof.* Let  $\mathcal{I}$  be an injective resolution of  $\mathcal{F}$ . Given a short exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

in  $\underline{\mathrm{Mod}}_X$ , we obtain the long exact sequence by taking  $\mathrm{Hom}$  or  $\mathcal{H}om$  into  $\mathcal{I}$ , yielding a short exact sequence of complexes (by the previous corollary), and then taking the long exact sequence of cohomology groups. One does need to check independence from the choice of the resolution, but this is similar to other arguments we've done before, so I won't bore you with it. (The summary: by a pushout construction, it suffices to compare  $\mathcal{I}$  and  $\mathcal{J}$  when there is a quasi-isomorphism  $\mathcal{I} \rightarrow \mathcal{J}$ . You then get a morphisms of short exact sequences which is a quasi-isomorphism on each term, etc.)  $\square$

Unfortunately, we can't check that  $\mathrm{Ext}^i(\cdot, \mathcal{F})$  and  $\mathcal{E}xt^i(\cdot, \mathcal{F})$  are effaceable, or construct them as derived functors, because  $\underline{\mathrm{Mod}}_X$  need not have enough *projectives* (exercise). However, we can still use certain "acyclic resolutions" to compute.

**Proposition.** Suppose that

$$\cdots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

is an exact sequence in  $\underline{\mathrm{Mod}}_X$ , where each  $\mathcal{L}_i$  is locally free of finite rank. (We say the  $\mathcal{L}_i$  form a locally free resolution of  $\mathcal{F}$ .) Then there is a isomorphism

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \cong h^i(\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G}))$$

which is functorial both in  $\mathcal{G}$  and in the resolution of  $\mathcal{F}$ .

*Proof.* Since  $\mathcal{L}_i$  is locally free of finite rank,  $\mathcal{E}xt^1(\mathcal{L}_i, \cdot)$  always vanishes, so  $\mathcal{H}om(\mathcal{L}_i, \cdot)$  is an exact functor (even though  $\mathrm{Hom}(\mathcal{L}_i|_U, \cdot)$  is not exact on any open  $U$ ). From this we see that the right side is a cohomological functor: given a short exact sequence  $0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G} \rightarrow \mathcal{G}_2 \rightarrow 0$ , the sequence

$$0 \rightarrow \mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G}_1) \rightarrow \mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G}) \rightarrow \mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G}_2) \rightarrow 0$$

is again exact, so admits a long exact sequence in cohomology.

We now have that both sides of the desired isomorphism are cohomological functors in  $\mathcal{G}$  whose higher terms vanish on injectives, so are effaceable. Hence they are both universal.  $\square$

Note that locally free resolutions are much easier to write down in practice than injective resolutions. For instance, if  $X = \mathbb{P}_k^n$  for  $k$  a field, and  $\mathcal{F}$  is coherent, then Serre's theorem gives a surjection  $\mathcal{E} \rightarrow \mathcal{F}$  where  $\mathcal{E}$  is a direct sum of twisting sheaves. Repeated application gives not just a locally free resolution but a free resolution!

**Proposition.** *For any coherent sheaves  $\mathcal{F}, \mathcal{G}$  on  $\mathbb{P}_k^n$  for  $k$  a field,  $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$  is again coherent.*

*Proof.* We just argued that a free resolution  $\mathcal{L}_\bullet$  of  $\mathcal{F}$  exists. By the previous proposition, we need only check that the  $h^i(\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G}))$  are coherent. This is true because the  $\mathcal{L}_\bullet$  and  $\mathcal{G}$  are coherent, so the  $\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G})$  are too.  $\square$

**Lemma.** *For  $\mathcal{F}, \mathcal{G}, \mathcal{L} \in \underline{\text{Mod}}_X$  with  $\mathcal{L}$  locally free of finite rank, there are canonical isomorphisms*

$$\text{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \text{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G})$$

and

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee.$$

*Proof.* Again, check that everything is an effaceable cohomological functor of  $\mathcal{G}$  and that things match at  $i = 0$ . (See Hartshorne Proposition III.6.7.)  $\square$

Final note: you may be wondering what the relationship is between  $\text{Ext}$  and  $\mathcal{E}xt$ . It comes from the following general fact: if  $F$  and  $G$  are left exact functors such that  $F \circ G$  makes sense, then there is a spectral sequence relating the derived functors of  $F$ , the derived functors of  $G$ , and the derived functors of  $F \circ G$ . (See Godement's book for details.) In our case, given a sheaf  $\mathcal{F}$ , take

$$F = H^0(X, \cdot), G = \mathcal{H}om_X(\mathcal{F}, \cdot), F \circ G = \text{Hom}_X(\mathcal{F}, \cdot).$$

## 2 Duality on projective space

For the rest of this lecture, we work over a field  $k$ , but it need not be algebraically closed.

**Theorem** (Duality on projective space). *Put  $X = \mathbb{P}_k^n$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Recall that  $H^n(X, \mathcal{O}_X(-n-1))$  is one-dimensional over  $k$ .*

(a) *The map*

$$\text{Hom}_X(\mathcal{F}, \mathcal{O}_X(-n-1)) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \mathcal{O}_X(-n-1))$$

*is a perfect pairing of finite dimensional  $k$ -vector spaces (i.e., it identifies each space with the Hom of the other into the target).*

(b) *For  $V$  a  $k$ -vector space, put*

$$V' = \text{Hom}_k(V, H^n(X, \mathcal{O}_X(-n-1))).$$

For each  $i \geq 0$ , there is a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{O}_X(-n-1)) \rightarrow H^{n-i}(X, \mathcal{F})'$$

which for  $i = 0$  reproduces (a), and which is compatible with short exact sequences.

*Proof.* For (a), we have a natural morphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X(-n-1)) \rightarrow H^n(X, \mathcal{F})'$$

of left exact covariant functors on  $\underline{\mathrm{Mod}}_X^{\mathrm{op}}$ , which we claim is an isomorphism. In case  $\mathcal{F} = \mathcal{O}_X(m)$ , we want a natural isomorphism

$$H^0(X, \mathcal{O}_X(-m-n-1)) \cong \mathrm{Hom}(H^n(X, \mathcal{O}_X(m)), H^n(X, \mathcal{O}_X(-n-1)))$$

and this is exactly what we got from Serre's calculation. Likewise, we already have the isomorphism when  $\mathcal{F}$  is a direct sum of twisting sheaves.

In general, we can write an exact sequence

$$\mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$$

in  $\underline{\mathrm{Mod}}_X$  with  $\mathcal{E}_0, \mathcal{E}_1$  both direct sums of twisting sheaves. Since the things we are computing are left exact on  $\underline{\mathrm{Mod}}_X^{\mathrm{op}}$ , this exact sequence turns into a diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X(-n-1)) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_0, \mathcal{O}_X(-n-1)) \longrightarrow \mathrm{Hom}(\mathcal{E}_1, \mathcal{O}_X(-n-1)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \sim \\ 0 & \longrightarrow & 0 & \longrightarrow & H^n(X, \mathcal{F})' & \longrightarrow & H^n(X, \mathcal{E}_0)' \longrightarrow H^n(X, \mathcal{E}_1)' \end{array}$$

The five lemma gives the desired isomorphism.

For (b), we have two cohomological functors on the category of coherent  $\mathcal{O}_X$ -modules which agree at index 0. We need only check that they are both effaceable. For this, given  $\mathcal{F}$  coherent, we can write it as a quotient of  $\mathcal{E} = \mathcal{O}_X(-q)^{\oplus m}$  for any sufficiently large  $q$ . So all we need to do is check that for any given  $i > 0$ , both  $\mathrm{Ext}^i(\mathcal{O}_X(-q), \mathcal{O}_X(-n-1))$  and  $H^{n-i}(X, \mathcal{O}_X(-q))$  vanish for  $q$  large. The second statement is true by Serre's calculation; so is the first because  $\mathrm{Ext}^i(\mathcal{O}_X(-q), \mathcal{O}_X(-n-1)) \cong H^i(X, \mathcal{O}_X(q-n-1))$ .  $\square$

### 3 Differentials and duality

This is not really the right way to view the duality theorem, because it does not generalize well. To fix this, we reintroduce the sheaf  $\Omega_{X/k}$  of Kähler differentials on  $X = \mathbb{P}_k^n$ , and its top exterior power  $\omega_X$ , the *canonical sheaf*.

**Lemma.** For  $X = \mathbb{P}_k^n$ , the sheaf  $\omega_X$  is isomorphic to  $\mathcal{O}_X(-n-1)$ .

*Proof.* This can be seen using the exact sequence

$$0 \rightarrow \Omega_{X/k} \rightarrow \mathcal{O}_X(-1)^{\oplus n+1} \rightarrow \mathcal{O}_X \rightarrow 0$$

of sheaves on  $X$ , where the middle term corresponds to the sheaf  $\oplus_{i=0}^n S(-1)e_i$ , the right term corresponds to  $S = k[x_0, \dots, x_n]$ , and the map  $S(-1)^{n+1} \rightarrow S$  takes  $e_i$  to  $x_i$  (Hartshorne, Theorem 8.13). This gives exact sequences

$$0 \rightarrow \Omega_{X/k}^i \rightarrow \wedge_k^i \mathcal{O}_X(-1)^{\oplus n+1} \rightarrow \Omega_{X/k}^{i-1} \rightarrow 0$$

for all  $i$ . For  $i = n+1$ , this becomes an isomorphism  $\mathcal{O}_X(-n-1) \rightarrow \Omega_{X/k}^{n+1}$  because  $\Omega_{X/k}^{n+1} = 0$ .

One can also see this more directly by writing down a global generator of  $\omega_X(n+1)$ . For instance, define  $\alpha \in H^0(D_+(x_0 \cdots x_n), \omega_X)$  by the formula

$$\begin{aligned} \alpha &= \frac{x_0^{n+1}}{x_0 \cdots x_n} d(x_1/x_0) \wedge \cdots \wedge d(x_n/x_0) \\ &= \frac{1}{x_0 \cdots x_n} \sum_{i=0}^n (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \end{aligned}$$

The first line shows that  $x_0 \cdots x_n \alpha$  generates  $\omega_X(n+1)$  over  $D_+(x_0)$ ; it also shows that performing an automorphism of  $X$  which swaps two of  $x_1, \dots, x_n$  only changes  $\alpha$  by a sign. The second line shows that the same is true of the automorphism of  $X$  which swaps  $x_0$  and  $x_n$ . Hence  $x_0 \cdots x_n \alpha$  generates  $\omega_X(n+1)$  over  $D_+(x_i)$  for  $i = 1, \dots, n$ .  $\square$

Warning: Hartshorne Remark 7.1.1 claims that  $\alpha$ , viewed as a Čech  $n$ -cocycle, is invariant under coordinate changes. However, we just contradicted this by showing that  $\alpha$  itself changes sign when you swap two coordinates. What is really happening is that if  $T : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$  is the linear automorphism defined by the matrix  $A$ , in the sense that

$$T^*(x_j) = \sum_i A_{ij} x_i \quad (i, j = 0, \dots, n),$$

then

$$T^*(x_0 \cdots x_n \alpha) = \det(A) x_0 \cdots x_n \alpha.$$

In any case, we can use  $\omega_X$  in place of  $\mathcal{O}_X(-n-1)$  in the statement of the duality theorem on projective space. Next time, I'll talk about how this can be generalized to other schemes over  $k$ .