18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Cohen-Macaulay schemes and Serre duality

In this lecture, we extend Serre duality to Cohen-Macaulay schemes over a field. As in the previous lecture, let k be a field (not necessarily algebraically closed), let $j : X \to P = \mathbb{P}^N_k$ be a closed immersion with X of dimension n, and let $\mathcal{O}_X(1)$ be the corresponding twisting sheaf.

1 Cohen-Macaulay schemes and duality

Let ω_X° denote a dualizing sheaf on X; remember that this choice includes a trace map $H^n(X, \omega_X^\circ) \to k$. We then obtain natural functorial maps

$$\theta^i : \mathrm{Ext}^i_X(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee$$

because both sides are cohomological functors on the opposite category of coherent sheaves on X, and the one on the left is effaceable because it vanishes on direct sums of twisting sheaves. By the definition of a dualizing sheaf, θ^0 is always an isomorphism.

Theorem. *The following conditions are equivalent.*

(a) The scheme X is equidimensional (each irreducible component has dimension n) and Cohen-Macaulay.

(b) The maps θ^i are isomorphisms for all $i \geq 0$ and all coherent sheaves $\mathcal{F}$ on X.

This is of course meaningless if I don't tell you what a Cohen-Macaulay scheme is. For the moment, suffice to say that a scheme is Cohen-Macaulay if and only if each of its local rings is a Cohen-Macaulay ring. That already has content, because then the theorem says that (b) is equivalent to a local condition on X, which is far from obvious.

I'll also point out that a regular local ring is always Cohen-Macaulay. This implies the following.

Corollary. *If X is smooth over k, then θ^i is an isomorphism for all $i \geq 0$ and all coherent sheaves $\mathcal{F}$ on X.*

2 Proof of the duality theorem, part 1

Even without knowing what a Cohen-Macaulay scheme is, we can at least start working to prove that condition (b) is equivalent to a *local* condition on X. Let us start by relating (b) to two global vanishing assertions.

Lemma. *The following conditions are equivalent to (b).*

(c) For any locally free coherent sheaf $\mathcal{F}$ on X, for q sufficiently large, we have $H^i(X, \mathcal{F}(-q)) = 0$ for all $i < n$.

(c′) For q sufficiently large, we have $H^i(X, \mathcal{O}_X(-q)) = 0$ for all $i < n$.

Note that condition (c) is a sort of opposite to Serre's vanishing theorem, which gives the vanishing of $H^i(X, \mathcal{F}(q))$ for $i > 0$ and q sufficiently large.

Proof. Given (b), for any locally free coherent sheaf $\mathcal{F}$ on X, we have

$$\begin{aligned} H^i(X, \mathcal{F}(-q)) &= \operatorname{Ext}_X^{n-i}(\mathcal{F}(-q), \omega_X^\circ)^\vee \\ &= \operatorname{Ext}_X^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ(q))^\vee \\ &= H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ(q))^\vee \end{aligned}$$

and this vanishes for $n - i > 0$ and q large by Serre's vanishing theorem. Thus (b) implies (c).

It is clear that (c) implies (c′). Given (c′), it follows that $H^{n-i}(X, \cdot)^\vee$ is effaceable for all $i > 0$ since we can cover $\mathcal{F}$ with a direct sum of twisting sheaves. Hence θ^i is the natural map between two universal cohomological functors, hence is an isomorphism. Thus (c′) implies (b). $\square$

We next reformulate this in local terms, using Serre duality on P.

Lemma. *The following condition is equivalent to (b).*

(d) For all $i < n$, $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$.

Remember that no matter what X is, we have $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$ for $i > n$: we proved this in the course of constructing the dualizing sheaf ω_X°.

Proof. By Serre duality on P (and choosing an isomorphism $H^n(P, \omega_P) \cong k$), we may identify

$$H^i(X, \mathcal{O}_X(-q)) \cong H^i(P, j_*\mathcal{O}_X(-q)) \cong \operatorname{Ext}_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q))^\vee.$$

So (c) is equivalent to the condition that for q sufficiently large, $\operatorname{Ext}_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q)) = 0$ for all $i < n$. Recall from earlier that for q large,

$$\operatorname{Ext}_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q)) = \Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q))) = \Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)(q)).$$

Since $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)$ is coherent, $\Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)(q))$ vanishes for q sufficiently large if and only if $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$. $\square$

Condition (d) can be rewritten as follows.

Lemma. *The following condition is equivalent to (b).*

(e) For each point $x \in X$, if $A = \mathcal{O}_{P,x}$ and I is the ideal of A defining X at x, then for all $i < n$, $\operatorname{Ext}_A^{N-i}(A/I, A) = 0$.

Proof. This translates directly from (d) once we remember that ω_P is locally free of rank 1 on P. $\square$

This is almost the local condition we are seeking, except that it still refers to the position of X within P.

3 The Cohen-Macaulay condition

To get rid of the dependence of our duality condition on the relative geometry of X within P, we need some more sophisticated commutative algebra.

Proposition. *Let A be a regular local ring and let M be a finitely generated A-module. Then for any nonnegative integer n, the following are equivalent.*

(a) We have $\mathrm{Ext}^i(M, A) = 0$ *for all* $i > n$.

(b) For any A-module N, we have $\mathrm{Ext}^i(M, N) = 0$ *for all* $i > n$.

(c) There exists a projective resolution $0 \to L_n \to \cdots \to L_1 \to L_0 \to M \to 0$ *of M at length at most n.*

Proof. See Hartshorne Proposition III.6.10A (and associated Matsumura reference) and exercise III.6.6. $\square$

The smallest integer for which this holds is called the *projective dimension* of M (if it exists), denoted $\mathrm{pd}_A(M)$. For instance, M is projective if and only if $\mathrm{pd}_A(M) = 0$.

For M a module over a ring A, a *regular sequence* is a sequence $x_1, \dots, x_n$ of elements of A such that for $i = 1, \dots, n$, x_i is not a zerodivisor on $M/(x_1, \dots, x_{i-1})M$. For A a local ring, the *depth* of M is the maximal length of a regular sequence with all x_i in the maximal ideal of A.

Proposition. *For A a regular local ring and M an A-module,*

$$\mathrm{pd}_A(M) + \mathrm{depth}_A(M) = \dim(A).$$

Proof. See Hartshorne Proposition III.6.12A (and associated Matsumura reference). $\square$

We can finally give a local equivalent to condition (b) from the duality theorem. Recall that our last equivalent (e) said that for each $x \in X$, for $A = \mathcal{O}_{P,x}$ and I the ideal of A defining X at x, $\mathrm{Ext}_A^{N-i}(A/I, A) = 0$ for all $i < n$. This is equivalent to $\mathrm{pd}_A(A/I) \leq N - n$, and hence to $\mathrm{depth}_A(A/I) \geq n$. The trick is that if M is an A/I-module, then $\mathrm{depth}_A(M) = \mathrm{depth}_{A/I}(M)$. Thus we have the following.

Lemma. *The following condition is equivalent to (b).*

(f) For each point $x \in X$, if $B = \mathcal{O}_{X,x}$, then $\mathrm{depth}_B(B) \geq n$.

On the other hand, we always have $\mathrm{depth}_B(B) \leq \dim(B) \leq n$, so it is equivalent to require $\mathrm{depth}_B(B) = \dim(B) = n$.

This condition $\mathrm{depth}_B(B) = \dim(B)$ is in fact the definition of a *Cohen-Macaulay* local ring B. Any regular local ring is Cohen-Macaulay, since we can use generators of the cotangent space as a regular sequence. But the Cohen-Macaulay condition is much more permissive; for instance, any *local complete intersection* is Cohen-Macaulay.