18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Sheaf cohomology (updated 13 Apr 09)

In the previous lecture, we discussed the construction of derived functors for left exact additive functors out of on an abelian category that has enough injectives. In this lecture, we specialize to the case of the global sections functor for sheaves on a locally ringed space, and thus obtain the definition of sheaf cohomology.

1 Having enough injectives

I thought I assigned this as homework, but apparently not, so here is the proof.

Lemma. *The category* $\underline{\text{Ab}}$ *has enough injectives.*

Proof. It has been assigned as an exercise that an abelian group G is injective if and only if it is *divisible*, i.e., if the multiplication by n maps are surjective for all positive integers n.

It remains to show that every group G is isomorphic to a subgroup of a divisible abelian group. For instance, write $G = F/H$ where F is a *free* abelian group, then embed G into $(F \otimes_{\mathbb{Z}} \mathbb{Q})/H$. If you want something more canonical, take F to be the free abelian group generated by the elements of G, with the map $G \to F$ taking each $g \in G$ to the generator of F indexed by g (a/k/a the adjunction morphism for the forgetful functor $\underline{\text{Ab}} \to \underline{\text{Set}}$). $\square$

There isn't quite as nice an argument for $\underline{\text{Mod}}_R$ because we don't have as simple a description of the injective modules. One proof that $\underline{\text{Mod}}_R$ has enough injectives is assigned as an exercise; another will be given using Grothendieck's criterion later in this lecture.

2 Categories of sheaves have enough injectives

Let X be a locally ringed space, let $\mathcal{C}$ be an abelian category, and let $\mathcal{D}$ be the category of sheaves on X with values in $\mathcal{C}$; then $\mathcal{D}$ is again an abelian category. However, in order to use the definition of derived functors, we need to know that $\mathcal{D}$ has enough injectives, i.e., that for any object $A \in \mathcal{D}$, there exists a monomorphism $A \to I$ with I injective. I should certainly assume that $\mathcal{C}$ itself has enough injectives; but then how can we go about constructing injective objects in $\mathcal{D}$?

One method is to try to identify the injective objects in $\mathcal{D}$, but that is a bit difficult, even for $\mathcal{C} = \underline{\text{Ab}}$. Another method is to construct a large enough class of injective objects using *skyscraper sheaves*. Let $x \in X$ be a point and let G be an object of $\mathcal{C}$. We may then view G as a sheaf on the one-point topological space $\{x\}$; the *skyscraper sheaf* at x with values in G, denoted $i_x(G)$ is the direct image of G along $\{x\} \to X$. Its sections are G on any open set containing x and 0 otherwise; its stalks are G at all points in the closure of x and 0 elsewhere.

If we assume that $\mathcal{C}$ has colimits, then we can use the adjointness property between direct and inverse image to assert that

$$\mathrm{Hom}_{\underline{\mathrm{Sh}}_{\mathcal{C}}(X)}(\mathcal{F}, i_x(G)) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_x, G).$$

In particular, if G is injective in $\mathcal{C}$, then $i_x(G)$ is injective in $\underline{\mathrm{Sh}}_{\mathcal{C}}(X)$. (Remember that this means that $\mathrm{Hom}(\cdot, i_x(G))$ is an exact functor.)

If we assume that $\mathcal{C}$ also has arbitrary products, it becomes easy to guess how to embed an arbitrary sheaf $\mathcal{F}$ into an injective: for each $x \in X$, use the hypothesis that $\mathcal{C}$ has enough injectives to construct a monomorphism $\mathcal{F}_x \to G_x$, and then $\mathcal{F}$ embeds into $\prod_{x \in X} i_x(G_x)$. Namely, for $U \subseteq X$ open, the map

$$\mathcal{F}(U) \to \left(\prod_{x \in X} i_x(G_x)\right)(U) = \prod_{x \in U} G_x = \prod_{x \in U} \mathcal{F}_x$$

takes a section s to the tuple (s_x) of its germs. This is a monomorphism by the sheaf axiom. Moreover, an arbitrary product of injective objects is injective.

In fact, something even stronger is true, and the proof is similar; see Hartshorne, Proposition III.3.2. (This reproduces the previous statement by taking the sheaf of rings to be a constant sheaf.)

Proposition. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

Beware that if X is a locally ringed space, it does not follow that the category of *quasicoherent* sheaves of $\mathcal{O}_X$-modules has enough injectives. (However, this is true for affine schemes because $\underline{\mathrm{Mod}}_R$ has enough injectives.)

3 More on having enough injectives

One can also establish that the category of sheaves has enough injectives using a very general criterion introduced by Grothendieck in *Sur quelques points...*

Theorem. *Let $\mathcal{C}$ be an abelian category satisfying the following conditions.*

(a) $\mathcal{C}$ admits arbitrary (small) direct sums.

(b) Suppose we are given a monomorphism $X \to Y$ in $\mathcal{C}$, a totally ordered set I, and an increasing family of subobjects Y_i of Y indexed by $i \in I$. (This last means that we are given a monomorphism $Y_i \to Y$ for each $i \in I$, and a monomorphism $Y_i \to Y_j$ for each i, j in I with $i \leq j$, such that $Y_i \to Y_j \to Y$ agrees with $Y_i \to Y$.) Then inside Y,

$$\left(\sum_i Y_i\right) \cap X = \left(\sum_i (Y_i \cap X)\right).$$

In other words, forming the direct limit of the Y_i commutes with taking the fibred product with X over Y. (The direct limits on both sides exist by (a).)

(c) *There exists an object $U \in \mathcal{C}$ such that for any monomorphism $X \to Y$ which is not an epimorphism, the map $\mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y)$ is also not an epimorphism. (That is, there is a map $U \to Y$ not factoring through X. Grothendieck calls U a* generator *of $\mathcal{C}$.) Also, the class of isomorphism classes of monomorphisms into U is small (this is automatic if $\mathcal{C}$ admits a forgetful additive functor to $\underline{\mathrm{Ab}}$).*

Then $\mathcal{C}$ has enough injectives.

Before proving this, I should point out that these conditions are sufficiently weak that they are satisfied by $\underline{\mathrm{Mod}}_R$. Namely, (a) and (b) are obvious, while (c) holds by taking $U = R$ because then $\mathrm{Hom}(U, \cdot)$ coincides with the forgetful functor to abelian groups. (It is also possible to prove more directly that $\underline{\mathrm{Mod}}_R$ has enough injectives, but never mind.)

I should also check a bit more carefully that these conditions are satisfied by the category of sheaves of abelian groups on a locally ringed space. To check (a), note that if $\mathcal{F}_i$ is a family of sheaves on X, then we may construct the direct sum by taking the sheafification of the presheaf $U \mapsto \oplus_i \mathcal{F}_i(U)$. We may check (b) stalkwise. To check (c), we take U to be the direct sum over open subsets $V \subseteq X$ of the pushforward $j_{V*}(\underline{\mathbb{Z}}_V)$ of the constant sheaf on V with values in $\mathbb{Z}$. The point is that for any sheaf $\mathcal{G}$,

$$\begin{aligned}
\mathrm{Hom}\left(\bigoplus_V j_{V*}(\underline{\mathbb{Z}}_V), \mathcal{G}\right) &= \bigoplus_V \mathrm{Hom}(j_{V*}(\underline{\mathbb{Z}}_V), \mathcal{G}) \\
&= \bigoplus_V \mathrm{Hom}(\underline{\mathbb{Z}}_V, \mathcal{G}|_V) \\
&= \bigoplus_V \Gamma(V, \mathcal{G}).
\end{aligned}$$

You can also use a direct sum over points, as in the previous section.

Lemma. *Under the conditions of the theorem, an object $M \in \mathcal{C}$ is injective if and only if for any monomorphism $V \to U$ into the generator, every morphism $V \to M$ extends to a morphism $U \to M$.*

Proof. Exercise. □

Proof of the theorem. We make a first approximation to the desired construction as follows. Let $M \in \mathcal{C}$ be any object. Let $I(M)$ be the set of isomorphism classes of pairs (T, t), where $T \to U$ is a monomorphism and $t : T \to M$ is a morphism. Consider the map

$$\oplus_{(T,t) \in I(M)} T \to M \times U^{I(M)}$$

in which the factor of T coming from a pair (T, t) maps to M via T, maps to the (T, t)-th factor of $U^{I(M)}$ via the monomorphism $T \to U$, and maps to the other factors of $U^{I(M)}$ via the zero map. Let $M \times U^{I(M)} \to C(M)$ be the cokernel of that map, and let $f(M) : M \to C(M)$ be the composition of this with the injection of M into the first factor of $M \times U^{I(M)}$. One checks using (b) that this is a monomorphism.

By construction, we have a monomorphism $f(M) : M \to I(M)$ such that for any monomorphism $T \to U$ and any morphism $T \to M$, we can extend $T \to M \to I(M)$ to a morphism $T \to I(M)$. This doesn't quite solve our problem because $M \neq I(M)$. The trick is to repeat this construction using transfinite induction. Namely, start with $M_0 = 0$. For any nonlimit ordinal i, put $M_{i+1} = f(M_i)$; for any limit ordinal, let M_i be the direct limit of M_j over $j < i$. There must then be a least ordinal k such that the cardinality of k is strictly greater than the cardinality of the number of isomorphism classes of monomorphisms into U. Then for any morphism $T \to M_k$, the sequence of inverse images of the M_j in T for $j < k$ must stabilize; that is, T maps into M_j for some M_j. Then this extends to a map of U into M_{j+1}, so M_k satisfies the condition of the previous lemma. □

4 Sheaf cohomology for topological spaces and ringed spaces

Let $\mathcal{C}$ be an abelian category admitting arbitrary products and colimits, and having enough injectives. We have just shown that for any topological space X, $\underline{\mathrm{Sh}}_{\mathcal{C}}(X)$ also has enough injectives. We may now define the *sheaf cohomology functors* $H^i : \underline{\mathrm{Sh}}_{\mathcal{C}}(X) \to \mathcal{C}$ to be the right derived functors of the left exact functor $\Gamma(X, \cdot) : \underline{\mathrm{Sh}}_{\mathcal{C}}(X) \to \mathcal{C}$. In particular, $H^0(X, \mathcal{F})$ is just another notation for $\mathcal{F}(X)$ or $\Gamma(X, \mathcal{F})$.

If $(X, \mathcal{O}_X)$ is a ringed space, we can also define derived functors of $\Gamma(X, \cdot)$ directly on the category of sheaves of $\mathcal{O}_X$-modules. The fact that these coincide with the H^i requires some justification, but it's not hard. One way to see it is to note that the H^i, when restricted to the category of $\mathcal{O}_X$-modules, return $\mathcal{O}(X)$-modules, then argue that these are an effaceable cohomological functor and so coincide with the derived functors.

Another argument is to use some acyclic objects which are not injective, remembering that we may use resolutions with these objects to compute derived functors. Here is a cheap supply of acyclic objects. A sheaf $\mathcal{F}$ on X is *flasque* (or *flabby*) if for any inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. For instance, if X is an *irreducible* topological space, then any constant sheaf is flasque. (Reminder: for $C \in \mathcal{C}$, the *constant sheaf* $\underline{C}_X$ on any space X is the sheafification of the constant presheaf $U \mapsto C$.) However, if $X = \mathbb{R}$ with the usual topology then the sections of $\underline{C}_X$ on X are C but on $\mathbb{R} \setminus \{0\}$ are $C \oplus C$, so $\underline{C}_X$ is not flasque unless $C = 0$.

Lemma. *For any ringed space $(X, \mathcal{O}_X)$, any injective $\mathcal{O}_X$-module is flasque. In particular (by taking $\mathcal{O}_X = \underline{\mathbb{Z}}_X$), any injective sheaf of abelian groups on X is flasque.*

Proof. (Compare Hartshorne, Lemma III.2.4.) Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. For any open subset U of X, let $\mathcal{O}_U$ denote the *extension by zero* of $\mathcal{O}_X|_U$ to X, i.e., the sheafification of the presheaf assigning V to $\mathcal{O}_X(V)$ if $V \subseteq U$ and 0 otherwise. Note that it has stalks $\mathcal{O}_{X,x}$ for $x \in U$ and 0 otherwise. (This differs from the direct image under the inclusion $U \hookrightarrow X$, which has nonzero sections on any open set *meeting* V.)

For $V \subseteq U$ an inclusion of open sets, we get a monomorphism $\mathcal{O}_V \to \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. Since $\mathcal{I}$ is injective, this gives a surjection $\mathrm{Hom}(\mathcal{O}_U, \mathcal{I}) \to \mathrm{Hom}(\mathcal{O}_V, I)$. But $\mathrm{Hom}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$ and $\mathrm{Hom}(\mathcal{O}_V, \mathcal{I}) = \mathcal{I}(V)$, so $\mathcal{I}$ is flasque. □

Proposition. *Let $\mathcal{F}$ be a flasque sheaf of abelian groups on a topological space X. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. The argument is a classic example of *dimension shifting*. Embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$, and put $\mathcal{G} = \mathcal{I}/\mathcal{F}$. Using the fact that $\mathcal{F}$ is flasque, we find (exercise)

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{I}) \to H^0(X, \mathcal{G}) \to 0$$

is exact. Using this, the long exact sequence in cohomology associated to

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0,$$

and the fact that $\mathcal{I}$ is acyclic, we find that $H^1(X, \mathcal{F}) = 0$ and

$$H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G}) \qquad (i > 1).$$

Since $\mathcal{F}$ is flasque, and $\mathcal{I}$ is injective and hence flasque by the previous lemma, it follows that $\mathcal{G}$ is flasque (exercise). Hence by the induction hypothesis, we may deduce $H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G}) = 0$ for $i > 1$. □

5 Sheaf cohomology and topological cohomology

If you know some topology, you might appreciate the following relationship between sheaf cohomology and the usual cohomology of topological spaces. (If not, pretend that the cohomology of the constant sheaf $\underline{\mathbb{Z}}_X$ is the definition of topological cohomology of a space X, then skip directly to the next section.)

Theorem. *Let X be a locally contractible topological space. Then the sheaf cohomology of X with coefficients in the constant sheaf $\underline{\mathbb{Z}}_X$ is canonically isomorphic to the singular cohomology of X.*

Recall that X is *contractible* if there is a continuous map $f : X \times [0, 1] \to X$ with $f(x, 0) = x$ for all $x \in X$, and $f(x, 1) = f(y, 1)$ for all $x, y \in X$; it is *locally contractible* if each point has a basis of contractible neighborhoods. For instance, all manifolds and CW-complexes are locally contractible.

The *singular n-chains* in X, collectively denoted $C_n(X)$, are formal finite $\mathbb{Z}$-linear combinations of continuous maps $\Delta : T_n \to X$, where T_n denotes the standard n-simplex. The *boundary map* $\delta : C_n(X) \to C_{n-1}(X)$ takes each simplex Δ to its signed boundary, i.e., if T_n has vertices $e_0, \ldots, e_n$, then for $i = 0, \ldots, n$, you take $(-1)^i$ times the restriction to the subsimplex omitting e_i. These form a homologically graded complex; putting

$C^n(X) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), \mathbb{Z})$ gives the *singular n-cochains*, which form a cohomologically graded complex.

Let $\mathcal{C}^n(X)$ be the sheafification of the presheaf $U \mapsto C^n(U)$; it is straightforward to check that in fact $\mathcal{C}^n(X)$ is flasque. Using the hypothesis that X is locally contractible (so that we can check exactness on stalks by running over a basis of contractible neighborhoods), one checks that

$$0 \to \mathcal{C}^0(X) \to \mathcal{C}^1(X) \to \cdots$$

is a resolution of $\underline{\mathbb{Z}}_X$. We may thus compute $H^i(\underline{\mathbb{Z}}_X)$ by computing global sections of this complex.

It remains to check that the natural map

$$C^{\cdot}(X) \to \Gamma(X, \mathcal{C}^{\cdot}(X))$$

is a quasi-isomorphism of complexes. To see this, let us fix an open cover $\{U_i\}$ of X, and let $D^{\cdot}(X)$ be the set of singular cochains only defined on simplices contained in some U_i. One then reduces to the following assertion.

Lemma. *The restriction $C^{\cdot}(X) \to D^{\cdot}(X)$ is a homotopy equivalence, with a quasi-inverse defined as follows. Given a cochain in $D^{\cdot}(X)$, extend to a cochain on X by mapping each simplex not contained in some U_i to 0.*

This is a standard if tedious calculation; see Spanier's *Algebraic Topology.*

6 Čech cohomology

From the previous section, we know that if X is a contractible topological space, then $\underline{\mathbb{Z}}_X$ is an acyclic sheaf (because the singular cohomology of X vanishes). This can be used to compute the cohomology of X in terms of the combinatorics of a *good cover*, i.e., an open cover $\{U_i\}$ of X in which each finite intersection is contractible. (You may have read about this in Bott and Tu, *Differential Forms in Algebraic Topology.*) We will use the same idea later in order to compute the cohomology of quasicoherent sheaves on schemes.

Let X be a topological space, and let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X (i.e., each $x \in X$ appears in only finitely many U_i). For convenience, let us assume the set I is equipped with a total ordering (this helps straighten out some sign conventions). For each finite subset J of I, put $U_J = \cap_{i \in J} U_i$, with the convention that $U_\emptyset = X$.

Let $\mathcal{F}$ be a sheaf of abelian groups on X. We define the *Čech complex* of $\mathcal{F}$ defined by the open cover $\{U_i\}$ as follows. For $j \geq 0$, let $\check{C}^j(\mathfrak{U}, \mathcal{F})$ be the direct product of $\Gamma(\mathcal{F}, U_J)$ over all $(j+1)$-element subsets J of I. The differential $d^j : \check{C}^j(\mathfrak{U}, \mathcal{F}) \to \check{C}^{j+1}(\mathfrak{U}, \mathcal{F})$ is defined as follows: for $\alpha = (\alpha_J) \in \check{C}^j(\mathfrak{U}, \mathcal{F})$, we have

$$d^j(\alpha)_J = \sum_{k=0}^{j+1} (-1)^k \operatorname{Res}_{U_{J-\{i_k\}}, J}(\alpha_{J-\{i_k\}}) \qquad J = \{i_0 \leq \cdots \leq i_{j+1}\}.$$

For instance, if there are only two open sets U_1 and U_2, then you have

$$0 \to \Gamma(\mathcal{F}, U_1) \oplus \Gamma(\mathcal{F}, U_2) \to \Gamma(\mathcal{F}, U_1 \cap U_2) \to 0$$

where the nontrivial map is the difference between the two restrictions. The signs were rigged up to make sure that this is indeed a complex: the point is that if you pull i_j and i_k out of a set J in on order and multiply the two resulting signs, you get the opposite sign as if you pulled them out in the opposite order.

It is an easy exercise to check that this gives a complex, and continues to do so if you insert $\Gamma(X, \mathcal{F})$ in front (with the individual restriction maps to $\check{C}^0(\mathfrak{U}, \mathcal{F})$.

It is convenient to also work with a sheafier analogue of this construction. Let $\check{\mathcal{C}}^j(\mathfrak{U}, \mathcal{F})$ be the direct product of $j_{J*}\mathcal{F}|_{U_J}$ over all $(j+1)$-element subsets J of I, where $j_J : U_J \to X$ is the inclusion. The global sections of this are just $\check{C}^j(\mathfrak{U}, \mathcal{F})$.

Lemma. *The complex*

$$0 \to \mathcal{F} \to \check{\mathcal{C}}^0(\mathfrak{U}, \mathcal{F}) \to \check{\mathcal{C}}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$$

is exact.

Proof. (Compare Hartshorne Lemma III.4.2.) It suffices to check exactness on stalks. Pick a point $x \in X$; we may then replace X by some U_i containing x. In this case, we can construct an explicit chain homotopy k between the identity map and the zero map. Its action can be described as follows: given a j-cochain $\alpha = (\alpha_J)$, you make a $(j-1)$-cochain by identifying α_J with a section of $U_{J \setminus \{i\}}$ whenever $i \in J$, and discarding the α_J whenever $i \notin J$. To do this correctly, you need to add some signs; I'll leave this to the Hartshorne reference. $\square$

We write $\check{H}^\cdot(\mathfrak{U}, \mathcal{F}) = h^\cdot(\check{C}^\cdot(\mathfrak{U}, \mathcal{F}))$. These do *not* form a cohomological functor if we fix the choice of $\mathfrak{U}$. As noted in Hartshorne Caution 4.0.2, this is clear for the trivial cover of X by itself because the global sections functor is not exact. However, they do at least give the right answer in the flasque case. (They also give the correct answer in degree 0 no matter what the cover, by the sheaf axiom!)

Lemma. *If $\mathcal{F}$ is flasque, then $\check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$ for $i > 0$.*

Proof. In the resolution

$$0 \to \check{\mathcal{C}}^0(\mathfrak{U}, \mathcal{F}) \to \check{\mathcal{C}}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$$

of $\mathcal{F}$, each term is again flasque and hence acyclic for sheaf cohomology. If we then take global sections and compute cohomology of the resulting complex, on one hand we just get $\check{H}^i(\mathfrak{U}, \mathcal{F})$. On the other hand, by the acyclic resolution theorem, we are also computing $H^i(X, \mathcal{F})$, which vanishes for $i > 0$. $\square$

On the other hand, suppose $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$, i.e., a new covering $\{V_j\}_{j \in J}$ equipped with a map $\lambda : J \to I$ of index sets such that $V_j \subseteq U_{\lambda(j)}$ for all $j \in J$. Then we get a restriction morphism

$$\check{H}^\cdot(\mathfrak{U}, \mathcal{F}) \to \check{H}^\cdot(\mathfrak{V}, \mathcal{F}).$$

Using refinements, the coverings of X form a direct system, so (since we are working with abelian groups, which admit colimits) we can form the direct limit

$$\check{H}^{\cdot}(X,\mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^{\cdot}(\mathfrak{U},\mathcal{F}).$$

Under certain circumstances, we can show that this computes sheaf cohomology. This won't cover the case of schemes, but we'll deal with that separately later.

Theorem. *Suppose that X is paracompact, i.e., X is Hausdorff and every open covering refines to a locally finite subcovering. Then the $\check{H}^{\cdot}(X,\mathcal{F})$ form a cohomological functor which is effaceable, hence universal, hence canonically isomorphic to $H^i(X,\mathcal{F})$. In particular, for any particular covering $\mathfrak{U}$, we obtain a morphism $\check{H}^{\cdot}(\mathfrak{U},\mathcal{F}) \to H^{\cdot}(X,\mathcal{F})$ functorial in $\mathcal{F}$.*

Proof. Since X is paracompact, we need only take the direct limit over locally finite coverings. In that case, the functors

$$\mathcal{F} \mapsto \varinjlim_{\mathfrak{U}} \check{C}^{\cdot}(\mathfrak{U},\mathcal{F})$$

are *exact* (exercise). Given that, we may apply them to a short exact sequence and then take the long exact sequence in cohomology to get the connecting homomorphisms. Effaceability holds because each $\mathcal{F}$ embeds into a sheaf which is injective, hence flasque, hence acyclic for $\check{H}^{\cdot}(X,\cdot)$ by an earlier lemma. □

All well and good, but what we really want to know is, when can we use the Čech complex associated to a particular complex $\mathfrak{U}$ to compute the cohomology of $\mathcal{F}$? Here is a useful answer in practice. We say the cover $\mathfrak{U}$ is *good for* $\mathcal{F}$ if for each J, $\mathcal{F}|_{U_J}$ is acyclic. (No hypothesis on X needed.)

Theorem (Leray)**.** *If $\mathfrak{U}$ is good for $\mathcal{F}$, then the morphisms $\check{H}^{\cdot}(\mathfrak{U},\mathcal{F}) \to H^{\cdot}(X,\mathcal{F})$ are isomorphisms. That is, the Čech complex $\check{C}^{\cdot}(\mathfrak{U},\mathcal{F})$ computes the sheaf cohomology of $\mathcal{F}$.*

Proof. As in the proof that Čech cohomology vanishes for flasque sheaves, it would suffice to show that the resolution

$$0 \to \check{\mathcal{C}}^0(\mathfrak{U},\mathcal{F}) \to \check{\mathcal{C}}^1(\mathfrak{U},\mathcal{F}) \to \cdots$$

is acyclic. Unfortunately, we can't directly conclude this from the fact that each $\mathcal{F}|_{U_J}$ is acyclic, because the direct image j_{J*} functor need not be exact.

So instead, we argue by dimension-shifting. The claim is evident for $i = 0$ by the sheaf axiom. Given the claim for all indices less than i, embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$, and let $\mathcal{G}$ be the quotient:

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

On each U_J, $\mathcal{F}$ and $\mathcal{I}$ are acyclic, so $\mathcal{G}$ is as well by the long exact sequence in cohomology. Moreover, we have short exact sequences

$$0 \to \Gamma(U_J,\mathcal{F}) \to \Gamma(U_J,\mathcal{I}) \to \Gamma(U_J,\mathcal{G}) \to H^1(U_J,\mathcal{F}) = 0.$$

This means that not only does this short exact sequence of sheaves give rise to a long exact sequence for the $H^i(X, \cdot)$, it also gives rise to a long exact sequence for the $\check{H}^i(\mathfrak{U}, \cdot)$ (because we get a short exact sequence of Čech complexes). We thus have a commuting diagram with exact rows:

$$\begin{tikzcd}
\check{H}^{i-1}(\mathfrak{U}, \mathcal{I}) \arrow[r] \arrow[d] & \check{H}^{i-1}(\mathfrak{U}, \mathcal{G}) \arrow[r] \arrow[d] & \check{H}^{i}(\mathfrak{U}, \mathcal{F}) \arrow[r] \arrow[d] & \check{H}^{i}(\mathfrak{U}, \mathcal{I}) \arrow[d] \\
H^{i-1}(X, \mathcal{I}) \arrow[r] & H^{i-1}(X, \mathcal{G}) \arrow[r] & H^{i}(X, \mathcal{F}) \arrow[r] & H^{i}(X, \mathcal{I})
\end{tikzcd}$$

in which the corners are zero (because injective implies flasque, which implies both the ordinary and Čech cohomologies vanish). So we transfer our question about $\mathcal{F}$ at index i to a question about $\mathcal{G}$ at index $i-1$, which we know by the induction hypothesis. □

This has practical applications outside of algebraic geometry: you can now use good covers to compute the singular cohomology of ordinary topological spaces! The analogue of this in algebraic geometry will appear next, when we start computing the cohomology of quasicoherent sheaves; the analogue of contractible open subsets in the topological case will turn out to be the *affine* schemes, on which quasicoherent sheaves will be acyclic.