18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Sheaves (updated 12 Feb 09)

We are now ready to introduce the basic building block in the theory of schemes, the notion of a *sheaf*. See also: Hartshorne II.1, EGA 1 0.3. (The latter means: look in the "Chapitre 0" section of EGA volume 1.) The base reference for this bit of EGA is Godement, *Théorie des Faisceaux*.

Note that Hartshorne assumes all sheaves take values in the category of abelian groups, that being the case of most interest in algebraic geometry. I will only impose that restriction in the next lecture.

1 Presheaves

Fix a category $\mathcal{C}$, e.g., sets or abelian groups. Given a topological space X, let $\underline{X}$ be the category of open sets of X. A *presheaf* on X with values in $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \underline{X} \to \mathcal{C}$. the category of open sets of X to $\mathcal{C}$. In other words, to specify a sheaf $\mathcal{F}$ on X, you must specify:

(a) for each open subset U of X an element $\mathcal{F}(U) \in \mathcal{C}$;

(b) for each inclusion $V \subseteq U$ of open subsets of X a morphism $\mathrm{Res}_{U,V} = \mathrm{Res}_{U,V}(\mathcal{F}) : \mathcal{F}(U) \to \mathcal{F}(V)$ (called *restriction*), such that:

  (i) for each open subset U of X, $\mathrm{Res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$;

  (ii) for each series of inclusions $W \subseteq V \subseteq U$ of open subsets of X, we have $\mathrm{Res}_{V,W} \circ \mathrm{Res}_{U,V} = \mathrm{Res}_{U,W}$ within $\mathrm{Hom}(U, W)$.

There seems to be some confusion over whether it is required that $\mathcal{F}(\emptyset)$ is required to be a final object of $\mathcal{C}$; Hartshorne's two characterizations of presheaves disagree on this point (because the definition of a functor doesn't include this condition). Fortunately, this doesn't have any serious consequences; the definition of a sheaf will stamp out this ambiguity. (EGA avoids this issue by omitting the definition of a presheaf entirely!)

We will typically use this definition in cases where $\mathcal{C}$ carries a forgetful functor to $\underline{\mathrm{Set}}$. In that case, it makes sense to speak of the elements of $\mathcal{F}(U)$ for U an open subset of X; we call these elements the *sections* of $\mathcal{F}$ on U. For $V \subseteq U$ an inclusion of open sets, and $s \in \mathcal{F}(U)$, we often write $s|_V$ instead of $\mathrm{Res}_{U,V}(s)$.

The *restriction* of a presheaf $\mathcal{F}$ on X to an open subset U of X is defined in the obvious fashion. It is denoted $\mathcal{F}|_U$. It is also called the *induced presheaf* of $\mathcal{F}$ on U.

If $\mathcal{F}_1, \mathcal{F}_2 : \underline{X} \to \mathcal{C}$ are both presheaves on a topological space X with values in a category $\mathcal{C}$, a *morphism* $\mathcal{F}_1 \to \mathcal{F}_2$ of presheaves is a natural transformation of functors from $\mathcal{F}_1$ to $\mathcal{F}_2$, i.e., a collection of maps $\mathcal{F}_1(U) \to \mathcal{F}_2(U)$ compatible with restrictions.

2 Sheaves

Here is an example of a set-valued presheaf $\mathcal{F}$: take another topological space Y, and put $\mathcal{F}(U) = \mathrm{Hom}_{\underline{\mathrm{Top}}}(U, Y)$ (the continuous functions from U to Y) with restriction being the usual restriction of functions. This example has a special feature not implied by the definition of a presheaf: a continuous function can be specified *locally*. In other words, for any index set I, if $\{V_i\}_{i \in I}$ is a family of open sets with union U, then on one hand, each element of $\mathcal{F}(U)$ is uniquely determined by its restrictions to all of the V_i; and on the other hand, any family of elements of $\mathcal{F}(V_i)$ which agree on the overlaps of the V_i gives a section over U.

This is formalized by the notion of a *sheaf*. A sheaf on X with values in $\mathcal{C}$ is a presheaf with the following property (called the *sheaf axiom*).

Axiom (Sheaf axiom)**.** For any index set I, for any family of open sets $\{V_i\}_{i \in I}$ which form a cover of the open set U, the object $\mathcal{F}(U)$ is the limit of the diagram formed by the $\mathcal{F}(V_i)$ for $i \in I$, the $\mathcal{F}(V_i \cap V_j)$ for $i, j \in I$, and the arrows $\mathrm{Res}_{V_i, V_i \cap V_j}$ for $i, j \in I$.

Let us make this explicit in case $\mathcal{C} = \underline{\mathrm{Set}}$. Define I, U, V_i as in the sheaf axiom.

(i) If $s_1, s_2 \in \mathcal{F}(U)$ is such that $s_1|_{V_i} = s_2|_{V_i}$ for all i, then $s_1 = s_2$. (If $\mathcal{C} = \underline{\mathrm{Ab}}$, we can just check this for $s_2 = 0$.)

(ii) Suppose we are given for each $i \in I$, an element $s_i \in \mathcal{F}(V_i)$ such that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Then there exists an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. (The element s is unique by (i).)

We define restriction of sheaves, and morphisms of sheaves, by copying the definitions from presheaves.

Some examples of sheaves:

- On a manifold, the continuous functions to some fixed topological space. Special example: if you take a target space C equipped with the discrete topology, you get what's called the *locally constant sheaf* associated to C.
- On a differentiable manifold, the differentiable functions.
- On a complex manifold, the holomorphic functions.
- On an abstract algebraic variety over an algebraically closed field, the regular functions, or the differential forms.

These all come from a class of objects called *locally ringed spaces*, which we will discuss later.

Although sheaves can be defined to take values in an arbitrary category, we will only be interested in cases where the category consists of objects with well-defined elements, and all the glueing is determined by the elements. So to keep things simple, let me drop in a hypothesis that I would like to keep in place from now on. (With only limits, Grothendieck calls this hypothesis (E). However, we'll want the colimits in order to talk about stalks later.)

Hypothesis (E)**.** Assume hereafter that all sheaves under discussion take values in a fixed category $\mathcal{C}$ which admits a forgetful functor to $\underline{\text{Set}}$ that *reflects small limits and colimits.* That is, all small (indexed by sets) limits exist, and their formation commutes with passage to $\underline{\text{Set}}$.

For example, $\mathcal{C}$ could be $\underline{\text{Set}}$ itself. It could also be any one of the usual "algebraic" categories: $\underline{\text{Ab}}$, $\underline{\text{Grp}}$, $\underline{\text{Ring}}$, $\underline{\text{Mod}}_R$ for a ring R, etc. Under this hypothesis, the sheaf axiom for $\mathcal{C}$ is exactly as for $\underline{\text{Set}}$, so a presheaf is a sheaf if and only if it becomes a sheaf after composing with the forgetful functor. We can thus forget the extra structure of $\mathcal{C}$ when checking basic facts about sheaves.

A typical bad example is $\underline{\text{Top}}$; the basic problem is that the image of a morphism under the forgetful functor can be an isomorphism even if the original morphism is not. That is, a continuous bijection of topological spaces need not be a homeomorphism.

Here is a trick for dealing with bad cases: given a presheaf $\mathcal{F}$ on X, for each object $Y \in \mathcal{C}$, let $\mathcal{F}_Y$ be the presheaf on X with values in $\underline{\text{Set}}$ defined by $U \mapsto \text{Hom}(Y, \mathcal{F}(U))$. Then $\mathcal{F}$ is a sheaf if and only if each $\mathcal{F}_Y$ is a sheaf.

3 Defining sheaves on a basis

It is very often convenient not to have to explicitly specify the sections of a sheaf on every open subset, but simply on a basis of open sets. Recall that a *basis (of open sets)* in a topological space X is a collection of open sets such that every open set can be written as a union of elements of the basis.

Let X be a topological space, and let $\underline{X}$ be the category of open sets of X. Let B be a basis of X, and let $\underline{B}$ be the full subcategory of $\underline{X}$ with $\text{Obj}(\underline{B}) = B$. (That is, keep all of the morphisms.) A *presheaf on X specified on B* is a contravariant functor from $\underline{B}$ to $\mathcal{C}$. A *sheaf on X specified on B* is a presheaf $\mathcal{F}$ on X specified on B, such that $\mathcal{F}$ satisfies the following modified sheaf axiom.

Axiom (Sheaf axiom for a basis)**.** For any index set I, for any $U \in B$ and any family of open sets $\{V_i\}_{i \in I}$ in B which form a cover of U, we can choose a covering $\{W_{ijk}\}_{k \in J_{i,j}}$ of each $V_i \cap V_j$ such that the object $\mathcal{F}(U)$ is the limit of the diagram formed by the $\mathcal{F}(V_i)$ for $i \in I$, the $\mathcal{F}(W_{ijk})$ for $i, j \in I$ and $k \in J_{i,j}$, and the arrows $\text{Res}_{V_i, W_{ijk}}$ for $i, j \in I$ and $k \in J_{i,j}$.

For example, suppose B is a basis in which the intersection of any two basic opens is a basic open; Ravi Vakil calls this a *nice* basis, so I will too. For a nice basis, this follows from the sheaf axiom applied to coverings of basic opens by other basic opens, because you just take the trivial covering of $V_i \cap V_j$ by itself. (The niceness condition is satisfied in most of our examples.)

Lemma (Basis lemma)**.** *Any sheaf on X specified on B extends uniquely to a sheaf on X. Similarly, any morphism between two sheaves on X specified on B extends to a morphism of sheaves on X.*

In other words, the restriction functor from sheaves on X to sheaves on X specified on B is an equivalence of categories.

Proof. Let $\mathcal{F}'$ be the presheaf defined by taking $\mathcal{F}(U)$ to be the limit of the diagram formed by the $\mathcal{F}(V)$ (and the restriction maps) for *all* basic opens V contained in U. If U is a basic open, then the construction comes with a map $\mathcal{F}'(U) \to \mathcal{F}(U)$ which defines a morphism of presheaves specified on B. Also, the limit property also defines the restriction maps $\mathrm{Res}_{U,V} : \mathcal{F}'(U) \to \mathcal{F}'(V)$ whenever $V \subseteq U$ are arbitrary opens, since $\mathcal{F}'(U)$ maps to $\mathcal{F}(W)$ for any basic open W contained in V. By a similar argument, any morphism $\mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism $\mathcal{F}' \to \mathcal{G}'$.

What is left to check that on one hand the map $\mathcal{F}'(U) \to \mathcal{F}(U)$ is an isomorphism, and on the other hand $\mathcal{F}'$ satisfies the sheaf axiom. We leave these as exercises. $\square$

As a corollary, we learn how to glue sheaves together.

Corollary. *Let I be an index set and let $\{U_i\}_{i \in I}$ be an open cover of X. Suppose we are given the following data.*

(a) For each $i \in I$, a sheaf $\mathcal{F}_i$ on U_i with values in $\mathcal{C}$.

(b) For each $i, j \in I$, an isomorphism $\theta_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \cong \mathcal{F}_j|_{U_i \cap U_j}$, satisfying the following conditions.

   (i) For each $i \in I$, θ_{ii} is the identity morphism on $\mathcal{F}_i$.

   (ii) For each $i, j, k \in I$, we have $\theta_{jk} \circ \theta_{ij} = \theta_{ik}$ as morphisms of sheaves on $U_i \cap U_j \cap U_k$. (This is called the cocycle condition*, for reasons to be discussed later.)*

Then there exist a sheaf $\mathcal{F}$ on X and isomorphisms $\theta_i : \mathcal{F}|_{U_i} \cong \mathcal{F}_i$ for each $i \in I$, such that for each $i, j \in I$, $\theta_{ij} \circ \theta_i = \theta_j$. Moreover, $\mathcal{F}$ is unique up to unique isomorphism (in a sense to be interpreted by the reader).

You might describe this by saying that "a sheaf of sheaves is a sheaf." In fact, this is the same sort of data needed to glue, say, topological spaces.

Proof. Suppose we are in the happy situation where whenever an open set U of X belongs to both V_i and V_j, we have a literal equality $\mathcal{F}_i(U) = \mathcal{F}_j(U)$ and the map θ_{ij} between these two is the identity morphism. (Note that the cocycle condition is automatically valid here.) Then we can apply the basis lemma, where B is the (nice) basis consisting of those open sets U contained in U_i for at least one index i.

The trouble is that as usual, objects in a category are usually not equal. However, using the cocycle condition we can force them to become equal as follows. Define a functor $\mathcal{F} : \underline{B} \to \mathcal{C}$ as follows. For $U \in B$, pick an index $i = i(U)$ such that $U \subseteq U_i$, and put $\mathcal{F}(U) = \mathcal{F}_i(U)$. For an inclusion $V \subseteq U$ of elements of B, put $i = i(U)$ and $j = i(V)$, so that V is contained in both U_i and U_j. Define $\mathrm{Res}_{U,V}(\mathcal{F})$ as the composition of the restriction map $\mathrm{Res}_{U,V}(\mathcal{F}_i) : \mathcal{F}_i(U) \to \mathcal{F}_i(V)$ with the map $\theta_{ij} : \mathcal{F}_i(V) \to \mathcal{F}_j(V)$. The cocycle condition then implies that these restriction maps are associative, so they define a presheaf $\mathcal{F}$ specified on B. The fact that each $\mathcal{F}_i$ is a sheaf implies that $\mathcal{F}$ is a sheaf specified on B, so it extends to a sheaf. $\square$

4 Stalks

An important source of information about sheaves is given by looking at their behavior "in the neighborhood of a point", as follows.

First let us recall something about direct limits. (Warning: I had the terminology slightly wrong when I introduced this in the category theory lecture. The notes have been corrected.) A *directed set* is a partially ordered set in which any two elements have an upper bound (but not necessarily a *least* upper bound). A *direct system* in a category $\mathcal{C}$ is a covariant functor $F : P \to \mathcal{C}$ with P a directed set. If the colimit exists, it is called the *direct limit* of the system.

Before using this notion for much, it might be helpful to make it explicit in the case of sets. (The case of abelian groups, which we also use, works the same way.) In this case, the direct limit is formed by starting with the union of $F(S)$ over all $S \in P$, then identifying the elements $x \in F(S)$ and $y \in F(T)$ if there exist arrows $f : S \to U$ and $g : T \to U$ in P such that $F(f)(x) = F(g)(y)$. A typical example is the formation of the fraction field $\mathrm{Frac}(R)$ of an integral domain R, as the direct limit of the rings $R[x]/(xf - 1)$ over all nonzero $f \in R$. Here the poset is the nonzero elements of R ordered under divisibily, and the map from $R[x]/(xf - 1)$ to $R[x]/(xfg - 1)$ takes x to xg.

Now let $\mathcal{F}$ be a presheaf on the topological space X, and let $x \in X$ be any point. View the open subsets of X containing x as a partially ordered set P_x under *reverse* inclusion. They then form a directed set, and the direct limit of the functor $\mathcal{F} : P_x \to \mathcal{C}$ is called the *stalk* of $\mathcal{F}$, denoted $\mathcal{F}_x$.

The elements of a stalk (which exist because we assumed (E)) are typically called *germs*. If s is a section of a sheaf on an open set containing x, we write s_x for the germ of s at x.

Example: the stalk of the sheaf of real-valued continuous functions consists of *germs* of real-valued continuous functions. Two continuous functions defined on open subsets of X containing a point x determine the same germ at x if and only if they coincide on some open subset containing x.

We can make a similar construction for the other "functions on manifolds" examples above. Beware that in these examples, the germ of a function at a point carries much more information than the *value* at that point. Extreme example: two holomorphic functions defined on a *connected* complex manifold have the same germ at a single point if and only if they coincide (because of analytic continuation!).

One variant we'll need a bit later: given *any* subset Z of X, not necessarily a single point, we can similarly take the direct limit of $\mathcal{F}(U)$ over all open subsets U of X containing Z. We call this the *stalk of X at Z*.

5 Stalks and morphisms

Stalks can be used to detect lots of interesting properties of sheaves, particularly in relation to morphisms. Throughout this section, let $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of sheaves on a topological space X.

Lemma. *Consider the following conditions.*

(a) For each $x \in X$, the map $\phi_x : \mathcal{F}_{1,x} \to \mathcal{F}_{2,x}$ is injective/surjective/bijective.

(b) For each open subset U of X, the map $\phi(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ is injective/surjective/bijective.

Then (b) implies (a) in all cases, while (a) implies (b) in the injective and bijective cases.

Proof. Suppose (a). Let Y_i be the product of $\mathcal{F}_{i,x}$ over all $x \in U$. Then the sheaf axiom implies that the map $\mathcal{F}_i(U) \to Y_i$ carrying s to $\prod_x s_x$ is injective. This gives injectivity in (b). (This is a toy example of the construction of the *espace étale* of a sheaf; I asked more about it on Problem Set 1.)

If ϕ_x is bijective for all x, then for any section $t \in \mathcal{F}_2(U)$ and any $x \in U$, there is an open neigborhood $V = V_x$ of x on which t coincides with the image under ϕ of some section $s_x \in \mathcal{F}_1(V_x)$. For $y \in U$ also, the restrictions of s_x and s_y to $\mathcal{F}_1(V_x \cap V_y)$ have the same image under ϕ (namely the restriction of t to $\mathcal{F}_2(V_x \cap V_y)$), so they coincide by what we proved in the previous paragraph. We can thus invoke the sheaf axiom to assemble $s \in \mathcal{F}_1(U)$ with $\phi(s) = t$. so surjectivity/bijectivity in (b) is an easy consequence.

Suppose (b). The surjectivity aspect is more or less obvious, so we only check the injectivity aspect. Suppose we are given two elements of $\mathcal{F}_{1,x}$ with the same image in $\mathcal{F}_{2,x}$. We can represent these by sections s_1, s_2 of $\mathcal{F}_1$ on some open neighborhood of x. In fact, we can take them on the same open neighborhood U. Their images are sections of $\mathcal{F}_2$ which have the same image in $\mathcal{F}_{2,x}$. That means that we can replace U by some smaller open neighborhood V so that $\phi(s_1)$ and $\phi(s_2)$ coincide in $\mathcal{F}_2(V)$. But then $s_1 = s_2$ in $\mathcal{F}_1(V)$, so (a) holds. $\square$

We define a morphism of sheaves to be *injective/surjective/bijective* if it has the corresponding property on stalks. By the previous lemma, bijective is the same as being an isomorphism (in the sense of having an inverse).

The disturbing thing is of course the failure of the implication from (a) to (b) in the surjectivity case. Yes, a morphism of sheaves can be surjective without being surjective on sections! What is true is: if ϕ is surjective and U is an open in X, then for each $s \in \mathcal{F}_2(U)$, we can cover U with open subsets V_i such that $\mathrm{Res}_{U,V_i}(s)$ is in the image of $\phi(V_i)$ for each i. The trouble is that you may not be able to choose elements of the $\mathcal{F}_1(V_i)$ which can be *glued*.

Here is a familiar example. Put $X = \mathbb{C} \setminus \{0\}$. Let $\mathcal{F}_1$ be the sheaf of holomorphic functions on X. Let $\mathcal{F}_2$ be the sheaf of *nowhere vanishing* holomorphic functions on X. Let $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ be the map taking $f : U \to \mathbb{C}$ to $\exp \circ f$. Then ϕ is surjective because the logarithm of a nonzero holomorphic function exists *locally*, but not globally: the function $z \in \mathcal{F}_2(X)$ is not in the image of $\phi(X)$.

6 Sheafification

If we fix a topological space X and a category $\mathcal{C}$, there is an obvious forgetful functor from sheaves on X with values in $\mathcal{C}$ to presheaves on X with values in $\mathcal{C}$. If you properly digested the notion of an adjoint functor, you should be asking whether this forgetful functor occurs as the right adjoint in an adjoint pair. It does!

Let $\mathcal{F} : \underline{X} \to \mathcal{C}$ be a presheaf on X with values in $\mathcal{C}$. Define another presheaf $\mathcal{F}^+$ on X as follows. For $U \subseteq X$ open, take $\mathcal{F}^+(U)$ to be the subset of $\prod_{x \in U} \mathcal{F}_x$ consisting of elements $s = \prod_x s_x$ with the following property: for each $x \in U$, there exists an open neighborhood V of x in U and a section $t \in \mathcal{F}(V)$ such that $s_y = t_y$ for all $y \in V$. From the definition, it is easy to check that $\mathcal{F}^+$ is a sheaf and that its stalk $\mathcal{F}^+_x$ is canonically isomorphic to $\mathcal{F}_x$. We call $\mathcal{F}^+$ the *sheafification* of $\mathcal{F}$; its construction is functorial in $\mathcal{F}$.

Proposition. *The functor $\mathcal{F} \mapsto \mathcal{F}^+$ from presheaves on X to sheaves on X, and the forgetful functor from sheaves on X to presheaves on X, form an adjoint pair.*

Proof. Exercise. □

7 Direct and inverse image

Let $f : X \to Y$ be a continuous map. For $\mathcal{F}$ a sheaf on X, the formula

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

obviously defines a sheaf $f_*\mathcal{F}$ on Y. It is called the *direct image* of $\mathcal{F}$.

Now let $\mathcal{G}$ be a sheaf on Y. Define a presheaf $f_-^{-1}\mathcal{G}$ on X as follows: for U open in X, let $(f_-^{-1}\mathcal{G})(U)$ be the stalk of $\mathcal{G}$ at $f(U)$, i.e., the direct limit of $\mathcal{G}(V)$ over open sets $V \subseteq X$ containing $f(U)$. This is general not a sheaf; its sheafification is called the *inverse image* of $\mathcal{G}$, denoted $f^{-1}\mathcal{G}$.

Proposition. *The functors f^{-1} and f_* form an adjoint pair.*

Proof. Exercise. □

You might wonder why I didn't use the notation f^* for the inverse image. That is because I will need that notation later for a different functor, defined for a morphism of ringed spaces.

Using the inverse image, we can define the *restriction* of $\mathcal{F}$ to an arbitrary subset Z of X, as the sheaf $i^{-1}\mathcal{F}$ for $i : Z \to X$ the inclusion map (with Z given the subspace topology). If $Z = \{x\}$, this coincides with the stalk $\mathcal{F}_x$ (exercise).