18.727: Topics in Algebraic Geometry, fall 2004
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About the course (last updated 3 May 04)

The topic for this course is *rigid analytic geometry*, which is an analogue of complex analytic geometry with the complex numbers replaced by a complete *nonarchimedean* field, such as the field of p-adic numbers $\mathbb{Q}_p$, or the field of Laurent series $k((t))$ over a field k.

Why?

Why would one ever want to construct such a theory? Here are a few examples where such ideas come up.

1. Rigid analytic geometry was first developed by Tate in order to construct the "Tate curve", a universal elliptic curve with bad reduction. This construction has become ubiquitous in the arithmetic theory of elliptic curves.

2. Raynaud used rigid geometry to resolve Abhyankar's conjecture, describing the possible Galois groups of unramified covers of the affine line over an algebraically closed field of positive characteristic. (No, it's not simply connected!) He also found a close relationship with the geometry of formal schemes.

3. Cherednik and Drinfeld constructed p-adic uniformizations of some symmetric spaces. This construction crept into number theory as part of Ribet's "level-lowering", the argument that explains why Wiles's work on the modularity of elliptic curves implies the Fermat conjecture. (p-adically uniformized varieties also provide instances where Deligne's weight-monodromy conjecture in étale cohomology can be verified, by work of de Shalit and Ito.)

4. The moduli space of Lubin-Tate formal group laws is a rigid analytic space occurring in the work of Gross and Hopkins, which ties together this sort of geometry with stable homotopy theory.

5. The work of Harris and Taylor on the local Langlands correspondence depends crucially on constructing good representations within the étale cohomology of rigid analytic spaces, building on a theory developed by Berkovich.

6. Berthelot's theory of rigid cohomology uses rigid geometry to construct a p-adic Weil cohomology which goes a long way towards filling the "gap at p" left by étale (ℓ-adic) cohomology.

7. How do you integrate a p-adic differential form (as opposed to a complex-valued function on a p-adic space)? The theory of p-adic integration, initiated by Coleman and pursued by Colmez, Berkovich, Besser and others, uses rigid-analytic techniques.

8. The classical theory of local fields associates a good ramification filtration to the Galois group of a complete discretely valued field with *perfect* residue field. Abbes and Saito have shown that you can do something similar for a general residue field (which is important when you consider families of varieties in positive characteristic), in terms of the action of the Galois group on certain rigid spaces.

9. The Lefschetz trace formula in étale cohomology for a self-correspondence on an algebraic geometry involves mysterious local contributions at the fixed point locus. Fujiwara has shown how to use rigid geometry to analyze these local contributions; for example, he verified a conjecture of Deligne that says these contributions become simple if you compose with a high enough power of Frobenius. (This is a key part of Lafforgue's work on the function field Langlands correspondence.)

10. The geometry of curves (and higher dimensional varieties) over the field $\mathbb{R}((t))$ is close to what is nowadays called *tropical algebraic geometry*, the study of the "algebraic geometry" of the semiring with operations max and $+$. This study yields interesting results about the topology of algebraic varieties over $\mathbb{R}$.

How?

How does one do analytic geometry over a field with a horribly disconnected topology?

Tate's original method was motivated by the construction of schemes, in which one builds geometric objects out of basic objects attached to rings. Tate realized that one could in principle construct complex analytic varieties by pasting together analytic subvarieties of polydiscs, and decided to use this as the method for building nonarchimedean analytic varieties.

To be precise, one replaces affine schemes by *affinoid varieties*, which amount to closed subvarieties of "closed polydiscs". One proves a lot of basic facts about these things that look like basic facts about subvarieties of affine spaces (i.e., about finitely generated algebras over a field).

Then one has to paste these affinoids together, but one must be a bit careful. One uses the formalism of Grothendieck topologies (i.e., knowing what "open covers" of a space are without an honest topology) to work with special types of covers (the so-called "admissible covers"). But in doing so, one recovers the ability to glue together coherent sheaves on affinoids and compute their cohomology on an admissible cover (theorems of Tate and Kiehl).

There is a second method for visualizing rigid analytic geometry due to Berkovich; we will introduce this after we get the Tate point of view up and running. It is based on the Gelfand-Mazur theorem that says that the points of a complex analytic space can be recovered as the norms on the algebra of functions on that space. Using an analogue of this construction on affinoid alegbras, then pasting together the results, gives somewhat peculiar topological spaces, but they are nice in certain ways: they are path-connected, unlike your typical p-adic topological space.

Where?

Where does one read about this?

There is basically one textbook on rigid analytic geometry: the book *Non-archimedean geometry* by Bosch, Güntzer, and Remmert. Unfortunately, it's not very useful as a text; it spends quite a lot of time on somewhat tangential issues of p-adic functional analysis, which are important for working with very general p-adic valued fields but can be treated much more simply in the case of a discretely valued field, where geometry tends to happen. As a result, the book doesn't manage to do any real geometry beyond constructing the Tate curve. Very unsatisfying.

As a result, I will be distributing detailed lecture notes for this course. My hope is to eventually assemble these notes into a book that does not suffer from the shortcomings I described in the previous paragraph.

Who?

Who is the target audience for this course?

This course is intended for students familiar with schemes and coherent sheaf cohomology, at the level of Chapters II and III of Hartshorne's book. (In particular, MIT's 18.726 or Harvard's Math 260 should be sufficient.) Since my interests are in arithmetic geometry, I will tend to incline towards arithmetic applications, but I'm open to suggestions of topics to discuss.

What?

What are the specific topics to be covered, and the requirements for students requiring a grade?

I'll be writing up separately a syllabus with particular topics; watch the web page for it.

For those students who need a grade for the coures, I will be giving occasional homework problems. I will probably also ask for a final paper (ideal length about 10 pages) on any topic you choose, as long as you can persuade me that it has something to do with the course material.

Your other principal responsibility is to find errors and inaccuracies in the notes and bring them to my attention. I tend to lose/forget information given to me in person, so I prefer such communications to happen via email. Even better still, you are free to grab the TeX files from my web site, make corrections yourself, and send the files back to me.

When?

When (and where) does the course meet?

See the course web site. I may have to cancel the occasional lecture due to an out-of-town trip, which I will try to make up at a mutually agreeable time.