18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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The G-topologies of an affinoid space

The goal of this lecture is to introduce some G-topologies on an affinoid space. Next time we'll prove Tate's theorem, which will establish the existence of the structure sheaf and of coherent sheaves of modules over the structure sheaf.

References: [FvdP, Section 4.1] and [BGR, 9.1.4]. Note that the *canonical topology* on $\operatorname{Max} A$ in [BGR] is not one of the G-topologies we're aiming for: it's just the topology induced by the supremum norm.

Affinoid subspaces

As usual, let K be a complete ultrametric field. Let A be an affinoid algebra over K and write $X = \operatorname{Max} A$. An *affinoid subspace* (or *affinoid subset*) of X is a subset $Y \subseteq X$ for which there exists a morphism $\phi : A \to B$ of affinoid algebras with $\phi(\operatorname{Max} B) \subseteq Y$, with the following universal property: given any morphism $\psi : A \to C$ of affinoid algebras with $\phi(\operatorname{Max} C) \subseteq Y$, there exists a unique morphism $\tau : B \to C$ with $\psi = \tau \circ \phi$. (I'll let you rewrite that in terms of the representability of an appropriate functor.) We will see shortly that ϕ is uniquely determined by Y; we call B the *coordinate ring* of Y.

The affinoid subspaces of X are analogues of the open affine subsets of an affine scheme (which obey an analogous universal property, though maybe you never noticed this before).

From [FvdP, Remarks 4.1.5], we collect the following observations.

Proposition 1. *Let Y be an affinoid subspace of X.*

(a) The map $\phi : A \to B$ is unique up to unique isomorphism.

(b) The induced map $\phi : \operatorname{Max} B \to Y$ is a bijection.

(c) For $y \in Y$, let $\mathfrak{m}_y$ and $\mathfrak{m}'_y$ are the maximal ideals of A and B, respectively, corresponding to y. Then the map $A/\mathfrak{m}_y^n \to B/(\mathfrak{m}'_y)^n$ is an isomorphism for each positive integer n.

(d) If Y is an affinoid subspace of X and Z is an affinoid subspace of Y, then Z is an affinoid subspace of X.

(e) If $\psi : A \to C$ is a morphism of affinoid algebras and Y is an affinoid subspace of $\operatorname{Max} A$ with coordinate ring B, $\psi(Y)$ is an affinoid subspace of $\operatorname{Max} B$ with coordinate ring $B \widehat{\otimes}_A C$. (The hat is missing in [FvdP, Remarks 4.1.5(4)].)

Proof. (a) is clear because the definition is via a universal mapping property. For (b) and (c), choose a point $y \in Y \subseteq X$ and an integer $n \geq 1$, and form the affinoid algebra $A/\mathfrak{m}_y^n$. Then the projection $A \to A/\mathfrak{m}_y^n$ factors uniquely through B, from which (b) and (c) follow. (d), (e), (f) are straightforward. $\square$

Corollary 2. *If $Y_1, \ldots, Y_n$ are affinoid subspaces of* $\operatorname{Max} A$ *with coordinate rings $B_1, \ldots, B_n$, then $Y_1 \cap \cdots \cap Y_n$ is an affinoid subspace with coordinate ring $B_1 \widehat{\otimes}_A \cdots \widehat{\otimes}_A B_n$.*

We may now define our first G-topology on $\operatorname{Max} A$. The *somewhat weak G-topology* on $\operatorname{Max} A$ is the G-topology in which admissible opens are affinoid subdomains, and admissible covers are covers containing a finite subcover. This (or rather, the one in which admissible covers are actually finite, but this is slightly finer than that) is the "weak G-topology" of [BGR], but it's a bit of a nuisance to prove anything about it. So following [FvdP], we are going to "sandwich" this topology between two others that yield the same topos.

Rational subspaces

Motivation: when proving the basic properties of schemes, one doesn't work with all affine opens. One restricts to the distinguished opens obtained by inverting elements, because those form a basis of the same topology. That's quite analogous to the way we are going to proceed here.

Let A be an affinoid algebra with $X = \operatorname{Max} A$. We say a subset Y of X is *rational* if there exist $f_0, \ldots, f_n \in A$ generating the unit ideal, such that

$$Y = \{x \in X : |f_i(x)| \leq |f_0(x)| \qquad i = 1, \ldots, n\}.$$

We now have the following result analogous to the structure theorem we proved on rational (affinoid) subsets of $\mathbb{P}$.

Proposition 3. *With notation as above, Y is an affinoid subspace of X with coordinate ring*

$$B = A\langle y_1, \ldots, y_n\rangle/(f_1 - f_0 y_1, \ldots, f_n - f_0 y_n).$$

Proof. Note that f_0 is a unit in B because $f_0, \ldots, f_n$ generate the unit ideal. Also, note that the obvious map $\phi : A \to B$ carries $\operatorname{Max} B$ into Y.

Let's now check the universal property for ϕ. Given $\psi : A \to C$ carrying $\operatorname{Max} C$ into Y, we must have $\psi(f_0) \in C^*$, and the spectral norms of the $\psi(f_i)/\psi(f_0)$ must be bounded by 1. We thus obtain a well-defined and unique morphism $\tau^* : A\langle y_1, \ldots, y_n\rangle \to C$ sending A to C via ψ and sending z_i to $\psi(f_i)/\psi(f_0)$ (see [FvdP, Proposition 3.4.7]). This map kills $f_i - f_0 y_i$ for each i, so factors uniquely through a map $\tau : B \to C$. Thus the universal property checks out. $\square$

The restriction that the f_i generate the unit ideal rules out such things as the subset of $\operatorname{Max} K\langle x, y\rangle$ on which $|x| \leq |y|$, for good reason: if you construct

$$K\langle x, y, z\rangle/(zx - y),$$

you get not the subset you want, but a blowup of it at the point $x = y = 0$.

Note that $Y = \emptyset$ if and only if $|f_0(x)| < \max_i\{|f_i(x)|\}$.

It may also be useful to note that $Y = \emptyset$ if and only if for some (or any sufficiently large) integer ℓ. For another equivalent form of this criterion, see the exercises.

I'll write $\mathcal{O}(Y)$ for the coordinate ring of Y. Define the *very weak G-topology* on X to be the one in which the admissible opens are rational subspaces, and the admissible covers are covers that include a finite subcover. Define the *weak G-topology* on X to be the one in which the admissible opens are finite unions of rational subspaces, and the admissible covers are again covers that include a finite subcover. Clearly the weak G-topology is slightly finer than the very weak G-topology.

We will show next time that every every affinoid subspace is a *finite* union of rational subspaces. ([FvdP] references the paper "Die Azyklizität der affinoiden Überdeckungen" by Gerritzen and Grauert; but it's also in [BGR, 8.2.2], which is where I'll take it from.) That means that on one hand, the somewhat weak G-topology is slightly finer than the very weak G-topology, but on the other hand the weak G-topology is slightly finer than the somewhat weak G-topology. So from the point of view of the sheaf theory, I can prove everything (like Tate's acyclicity theorem) using the very weak G-topology, where it's much easier.

Incidentally, it is possible to write down affinoid subspaces which are not rational; see [FvdP, Exercise 4.1.6] for one example.

Exercises

1. Let A be an affinoid algebra. Prove that the rational subspace of A defined by $f_0, \ldots, f_n$ is empty if and only if for any sufficiently large positive integer ℓ, there exists an expression
$$f_0^\ell = \sum_\alpha c_\alpha f_1^{\alpha_1} \cdots f_n^{\alpha_n}$$
of f_0^ℓ as a homogeneous polynomial of degree ℓ in $f_1, \ldots, f_n$ with coefficients in $\mathfrak{m}_K$. (Hint: see [FvdP, Proposition 4.1.2(4).])