18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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Affinoid algebras and their spectra

Last time we discussed Tate algebras and defined affinoid algebras to be the quotients of Tate algebras. This time, we'll begin the process of turning those algebras into "spaces" by studying their maximal spectra. Of course, this is not the right way to go in the long run; this is analogous to constructing varieties, whereas one should be doing something more "schematic". We'll return to this point when we discuss Berkovich spaces.

Warning: I dashed these notes off in a bit of a hurry, so there are probably lots of mistakes, which we'll doubtless find in class. As usual, if you send me corrections by email, I'll change the file on the web accordingly. Sorry about that.

References: [FvdP, Sections 3.3 and 3.4], [BGR, Section 3.8] (and various other sections of BGR which you'll find via the index, sigh).

Review: Newton polygons

Probably you've all seen this before, but just in case, let me review the theory of the Newton polygon of a polynomial over an ultrametric field. (I really should have done this back in the ultrametric field section.)

Lemma 1. *Let L be a complete ultrametric field, and let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial over L with roots $\alpha_1, \dots, \alpha_n \in L^{\mathrm{alg}}$ (repeated with appropriate multiplicities). Sort the roots so that $|\alpha_1| \geq |\alpha_2| \geq \cdots |\alpha_n|$. Then for $i = 1, \dots, n$,*

$$|a_{n-i}| \leq |\alpha_1 \cdots \alpha_i|,$$

with equality whenever $|\alpha_i| > |\alpha_{i+1}|$, or when $i = n$.

Proof. Write $P(z) = \prod_{i=1}^n (z - \alpha_i)$. Then a_{n-i} is, up to sign, the sum of the i-fold products of the α's, so the desired inequality is clear. As for the equality, note that if $|\alpha_i| > |\alpha_{i+1}|$, then $\alpha_1 \cdots \alpha_i$ is the unique i-fold product of maximum norm. □

Corollary 2. *The maximum of $|\alpha|$ over the roots of P is equal to $\max_j\{|a_{n-j}|^{1/j}\}$.*

If you want all of the absolute values of the roots, you get them from the lemma as follows. Consider the set of points $(n - i, -\log a_i)$ for $i = 0, \dots, n$ (where $a_n = 1$ by convention). Form the lower convex hull of this set of points: the lower boundary of this hull is called the *Newton polygon* of P. If r occurs as a slope in this polygon of a segment of width j, then there are exactly j roots of P of norm $\exp(r)$.

The maximal spectrum and the spectral seminorm

Let A be an affinoid algebra, and let $\operatorname{Max} A$ denote the set of maximal ideals of A, a/k/a the *maximal spectrum* of A. We refer to $\operatorname{Max} A$ as the *affinoid space* associated to A, although for the time being it's just a set; we'll give it a "topology" and a ringed space structure later.

By the Nullstellensatz for Tate algebras (proved last time), for each $\mathfrak{m} \in A$, $A/\mathfrak{m}$ is finite dimensional over K; in particular, it admits a unique extension of the norm on K. We will use function-theoretic notation to speak about $\operatorname{Max} A$; that is, if x is a "point" of $\operatorname{Max} A$ corresponding to the maximal ideal $\mathfrak{m}_x$ and $f \in A$, we will write $f(x)$ to mean the image of f in $A/\mathfrak{m}_x$.

Define the *spectral seminorm* of A (or $\operatorname{Max} A$) as

$$\|f\|_{\mathrm{spec}} = \sup_{x \in \operatorname{Max} A} |f(x)|.$$

(The term *supremum seminorm* is used interchangeably, as in [BGR].) It is straightforward to check that $\|\cdot\|_{\mathrm{spec}}$ is actually a seminorm, and that $\|fg\|_{\mathrm{spec}} \leq \|f\|_{\mathrm{spec}}\|g\|_{\mathrm{spec}}$. The spectral seminorm is a norm if and only if the intersection of the maximal ideals of A is the zero ideal; in that case, we also call it the *spectral norm*. It will turn out that this happens if and only if A is reduced; see below.

Note that from what we showed last time, the spectral norm on T_n is precisely the Gauss norm, and the supremum defining the spectral norm is achieved at some point. (More precisely, last time we showed that if K has infinite residue field, the supremum is actually achieved at some K-rational point. But the same argument shows that for any K, the supremum is achieved at some point defined over a finite extension of K.) That this holds in general is the content of the "maximum modulus principle" for affinoid spaces; see below.

Proposition 3 (Maximum modulus principle). *For A an affinoid algebra and $f \in A$, there exists $x \in \operatorname{Max} A$ such that $\|f\|_{\mathrm{spec}} = |f(x)|$. Moreover, if $\|f\|_{\mathrm{spec}} = 0$, then f is nilpotent.*

Proof. There is no harm in quotienting A by its nilradical, since computing $|f(x)|$ is insensitive to nilpotents. That is, we may assume A is reduced. We may also assume A is integral, since otherwise we can check the claim on each connected component of A.

So assume that A is an integral domain. By Noether normalization, A can be written as a finite integral extension of some T_d. That means there is an irreducible polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ over T_d such that $P(f) = 0$. (Note that the coefficients of P lie in T_d and not its fraction field, because T_d is a unique factorization domain and so "Gauss's lemma" applies.) From the theory of the Newton polygon, for any $x \in \operatorname{Max} A$ lying over $y \in \operatorname{Max} T_d$, we have $|f(x)| = \max_i\{|a_{n-i}(y)|^{1/i}\}$. Running this over all x and y, we get

$$\|f\|_{\mathrm{spec}} = \max_i\{\|a_{n-i}\|_{\mathrm{spec}}^{1/i}\},$$

and the maximum on the right is achieved because we already know the maximum modulus principle for T_d. This proves the first claim. As for the second claim, note that $\|f\|_{\mathrm{spec}} = 0$ implies $\|a_{n-i}\|_{\mathrm{spec}} = 0$ for all i. Since the spectral seminorm on T_d is a norm, we have $a_{n-i} = 0$ for all i, and so f is nilpotent. $\square$

Corollary 4. *For A an affinoid algebra, the intersection of the maximal ideals of A equals the nilradical of A. (That is, A is a "Jacobson ring".) In particular, the spectral seminorm is a norm if and only if A is reduced.*

Let $\mathfrak{o}_A^{\text{spec}}$ be the subring of A consisting of those $f \in A$ for which $|f|_{\text{spec}} \leq 1$.

Lemma 5. *For $\phi : A \to B$ a finite injective homomorphism of affinoid algebras and $f \in A$, one has $\|f\|_{\text{spec}} = \|\phi(f)\|_{\text{spec}}$.*

Proof. The finiteness of ϕ means that the map $\operatorname{Max} B \to \operatorname{Max} A$ is surjective, from which the claim follows. □

Lemma 6. *Suppose $\phi : T_d \to A$ is a finite injective K-algebra homomorphism. Then $\mathfrak{o}_A^{\text{spec}}$ is integral over $\phi(\mathfrak{o}_{T_d}^{\text{spec}})$.*

Note that $\phi(\mathfrak{o}_{T_d}^{\text{spec}}) \subseteq \mathfrak{o}_A^{\text{spec}}$ by the previous lemma, so the statement makes sense.

Proof. If A is an integral domain, the proof of the maximum modulus principle yields that any $f \in \mathfrak{o}_A^{\text{spec}}$ is the root of a polynomial over $\phi(\mathfrak{o}_{T_d}^{\text{spec}})$. For the reduction to this case, see [FvdP, Proposition 3.4.5]. □

Lemma 7. *For A an affinoid algebra under some norm $\|\cdot\|$ and $f \in A$, one has $\|f\|_{\text{spec}} \leq 1$ if and only if the sequence $\{\|f^n\|\}_{n=1}^{\infty}$ is bounded.*

Proof. If $\|f^n\|$ is bounded, then $|f^n(x)|$ is bounded for each $x \in \operatorname{Max} A$, so $|f(x)| \leq 1$, and so $\|f\|_{\text{spec}} \leq 1$. Conversely, suppose $\|f\|_{\text{spec}} \leq 1$. Write A as a finite integral extension of some T_d. By the previous lemma, f is integral over $\mathfrak{o}_{T_d}^{\text{spec}}$, i.e., it is the root of some $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with $a_i \in T_d$ and $\|a_i\|_{\text{spec}} \leq 1$. Each power of f can be written as a linear combination of $1, f, \dots, f^{n-1}$ whose coefficients are polynomials in the a_i with integer coefficients. The set of such polynomials is bounded under the Gauss norm in T_d, so by the continuity of the map $T_d \to A$, it is also bounded under the norm on A. Thus the powers of f are bounded. □

The following corollary of the previous proposition might have been what you were expecting when I first said "spectral seminorm".

Corollary 8. *Let A be an affinoid algebra with norm $\|\cdot\|$. Then for $f \in A$,*

$$\|f\|_{\text{spec}} = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

More on spectral seminorms in the next handout.

Exercises

1. Suppose A is an affinoid algebra and $f \in A$. Prove that the following are equivalent:
   (a) $\inf\{|f(x)| : x \in \operatorname{Max} A\} > 0$;
   (b) $f(x) \neq 0$ for all $x \in \operatorname{Max} A$;
   (c) $f \in A^*$.

(Note: this is exactly [FvdP, Exercise 3.3.4(1).])

2. Suppose $\rho_1, \ldots, \rho_n \in (0, \infty)$ are such that some power of each ρ_i belongs to $|K^*|$. Prove that the "modified Tate algebra"

$$T_{n,\rho} = \{\sum_I c_I x^I \in K[\![x_1, \ldots, x_n]\!] : |c_I|\rho_1^{i_1} \cdots \rho_n^{i_n} \to 0\}$$

is an affinoid algebra. (See [FvdP, Exercise 3.3.4(5)].)

3. Suppose $\rho \in (0, \infty)$ is such that no power of ρ belongs to $|K^*|$. Prove that $T_{1,\rho}$ is *not* an affinoid algebra. (Hint: what is the spectral norm of t_1?)