18.727, Topics in Algebraic Geometry (rigid analytic geometry)
Kiran S. Kedlaya, fall 2004
Berkovich spaces for dummies

References: For the little I'm going to say today, [FvdP, Section 7.1] is a sufficient reference. (Note I'm using their "filters" instead of Berkovich's "nets".) For more details, you have to read Berkovich, which can be a daunting task. Definitely start with his ICM 1998 talk ("p-adic analytic spaces"), then try his IHES paper "Étale cohomology for non-Archimedean analytic spaces". Beware that his earlier monograph "Spectral theory and analytic geometry over non-Archimedean fields" makes some definitions which are not consistent with the later papers (or so Johan tells me, I only looked at the later papers).

Addenda: analytification and properness

A couple of comments from Brian Conrad, clarifying some points from earlier.

Re analytification: SGA1, Exposé XII is worth a look for the complex analytic setup; basically all the formalism there goes through without incident. (For more comments specifically on the rigid case, see section 5 of Brian's paper "Irreducible components of rigid spaces", *Ann. Inst. Fourier (Grenoble)* **49** (1999), 473–541; but be forewarned that basically that will tell you what I just said.) Abstractly, the analytification of a K-scheme X locally of finite type can be characterized by the fact that it represents the functor associating to an analytic space Y the set of ringed space maps $Y \to X$. That means it's unique if it exists; existence is a series of reductions to affine space.

Unfortunately, dealing with properness seems to really be as hard as we were finding it to be in class. Brian's paper deals with why properness for an algebraic variety is equivalent to properness for its analytification; it uses comparatively recent work of Temkin (student of Berkovich); in particular, proving that the composition of proper maps is proper is a big hassle.

Re GAGA: the GAGA theorems are also true in the proper case (I forget who pointed that out), and you can see the reduction to the projective case (using Chow's lemma) in SGA1, Exposé XII, section 4.

Filters (again)

I snuck filters onto a previous handout, but since I didn't go over them in class, I'd better repeat myself.

Given a space X equipped with a G-topology, a *G-filter* (or simply "filter") on X is a collection $\mathcal{F}$ of admissible subsets with the following properties.

(a) $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.

(b) If $U_1, U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$.

(c) If $U_1 \subseteq U_2$ and $U_1 \in \mathcal{F}$, then $U_2 \in \mathcal{F}$.

A *prime filter* is a filter $\mathcal{F}$ also satisfying

(d) if $U \in \mathcal{F}$ and $\{U_i\}_{i\in I}$ is an admissible covering of U, then $U_i \in \mathcal{F}$ for some $i \in I$.

A *maximal filter* (or *ultrafilter*) is a filter $\mathcal{F}$ which is maximal under inclusion; such a filter is clearly also prime. For each $x \in X$, the set of admissibles containing x is a maximal filter.

Let $\mathcal{P}(X)$ and $\mathcal{M}(X)$ denote the sets of prime and maximal filters, respectively, on X, and likewise for any admissible open U of X (that is, $\mathcal{P}(U)$ consists of prime filters of X in which U appears). Equip $\mathcal{P}(X)$ with the ordinary topology generated by the $\mathcal{P}(U)$; then there is a natural morphism of sites $\sigma : X \to \mathcal{P}(X)$, and it turns out that the functors σ_* and σ^* are equivalences between the categories of abelian sheaves on X and on $\mathcal{P}(X)$ [FvdP, Theorem 7.1.2].

Moreover, $\mathcal{P}(X)$ has "enough points" in the topos-theoretic sense: you can check whether a sheaf is zero by checking that its stalks at points of $\mathcal{P}(X)$ are all zero. That's cold comfort if you can't get a handle on those stalks, but for rigid spaces you can!

Filters and valuations

The spaces $\mathcal{P}(X)$ and $\mathcal{M}(X)$ are pretty unwieldy in general, but for rigid analytic spaces, we can make them more explicit.

First of all, let X be a rigid space. Then X carries a structure sheaf $\mathcal{O}$. That sheaf has a subsheaf of rings $\mathfrak{o}$, consisting of functions of spectral seminorm bounded by 1 everywhere. (That is, if U is an affinoid subspace and $A = \Gamma(\mathcal{O}, U)$, then $\Gamma(\mathfrak{o}, U) = \mathfrak{o}_{A,\mathrm{spec}}$.)

Now let $p \in \mathcal{P}(X)$ be a prime filter. Let $\mathcal{O}_p$ and $\mathfrak{o}_p$ be the stalks of $\mathcal{O}$ and $\mathfrak{o}$, respectively, at p; that is, $\mathcal{O}_p = \lim_{\to} \Gamma(\mathcal{O}, U)$ for U running over affinoid subsets U of X containing p, and similarly for $\mathfrak{o}_p$. (You may of course run the limit over any cofinal set of neighborhoods, e.g., rational subsets of a particular affinoid neighborhood.) Define the seminorm $\|\cdot\|_p$ on $\mathcal{O}_p$ by

$$\|f\|_P = \inf\{\|f\|_U : U \in p, f \in \Gamma(\mathcal{O}, U)\}.$$

Let $\mathfrak{m}_p$ be the ideal of $\mathcal{O}_p$ consisting of elements of seminorm 0; of course $\mathfrak{m}_p \subseteq \mathfrak{o}_p$ also.

To speak intelligently about this stalk, we need a bit of valuation theory. Let G be a divisible totally ordered group; then I can view G as a vector space over $\mathbb{Q}$. The completion $\widehat{G}$ of G is then a vector space over $\mathbb{R}$; its dimension is the *real rank* of G. If G is a totally ordered group but not divisible, define its real rank to be the real rank of its divisible closure. If R is a valuation ring, define the real rank of R to be the real rank of its valuation group $(\mathrm{Frac}\, R)^*/R^*$ (viewed additively, contrary to our usual convention). Note that R has real rank 1 if and only if its valuation corresponds to a nonarchimedean valuation $|\cdot| : (\mathrm{Frac}\, R)^* \to \mathbb{R}_{>0}$.

We now state [FvdP, Proposition 7.1.8]. It's enough to consider affinoid spaces, since the stalk only depends on an affinoid neighborhood.

Proposition 1. *Let $X = \mathrm{Max}(A)$ be an affinoid space and let p be a prime filter of X.*

(a) The ring $\mathcal{O}_p$ is a henselian local ring with maximal ideal $\mathfrak{m}_p$. (We will hereafter denote its residue field by k_p.)

(b) *The ring* $\mathfrak{o}_{k_p} = \mathfrak{o}_p/\mathfrak{m}_p$ *is a valuation ring with fraction field* k_p*, and its real rank is at most* $\dim(X) + 1$*. Also, if* $\pi \in K$ *satisfies* $0 < |\pi| < 1$*, then* $\cap_{n=1}^{\infty} \pi^n \mathfrak{o}_{k_p} = 0$*.*

(c) *Let* $\mathfrak{p}$ *be the kernel of* $A \to k_p$*, and let* B *be the inverse image of* $\mathfrak{o}_{k_p}$ *in* $\operatorname{Frac} A/\mathfrak{p}$*. Then the image of* $\mathfrak{o}_{A,\mathrm{spec}}$ *in* $A/\mathfrak{p}$ *is contained in* B*; moreover,* $\cap_{n=1}^{\infty} \pi^n B = 0$*.*

Proof. See [FvdP, Proposition 7.1.8]. □

We thus have a meaningful valuation associated to any prime filter; the converse is also true. More precisely, a *valuation on* X consists of a pair $(\mathfrak{p}, B)$, in which $\mathfrak{p}$ is a prime ideal of A and B is a valuation ring of $\operatorname{Frac} A/\mathfrak{p}$ with the following properties.

(a) The image of $\mathfrak{o}_{A,\mathrm{spec}}$ in $\operatorname{Frac} A/\mathfrak{p}$ lies in B.

(b) For some (any) $\pi \in K$ with $0 < |\pi| < 1$, $\cap_{n=1}^{\infty} \pi^n B = 0$.

Proposition 2. *(a) The construction of Proposition 1 yields a bijection between the prime filters on* X *and the valuations on* X*.*

(b) Under the bijection in (a), the maximal filters on X *correspond precisely to the valuations of real rank 1.*

Proof. See [FvdP, Theorem 7.1.10]. □

Time out: Gelfand-Naimark

It is worth being reminded of a fundamental fact from classical functional analysis, that will put what we just did in a better context and suggest how to move forward.

Let X be a compact (Hausdorff) topological space, and let $C(X)$ denote the space of $\mathbb{C}$-valued continuous functions on X. Then X is a commutative C^*-algebra under the supremum norm (i.e., it's complete, it carries a complex conjugation $*$, and you can compute the norm of f as the square root of the norm of ff^*). The points of X carry algebraic meaning in $C(X)$: they give rise to maximal ideals on $C(X)$ (which are all distinct by Hausdorffness).

The Gelfand-Naimark theorem (which algebraic geometers might think of as the analogue of the Nullstellensatz in this context) asserts that on one hand these are all the maximal ideals of X, and on the other hand *any* commutative C^*-algebra A can be realized as $C(X)$ by putting a suitable topology on $X = \operatorname{Max} A$. Namely, for $x \in A$, $A/\mathfrak{m}_x$ is isomorphic to its subring $\mathbb{C}$ by the Gelfand-Mazur theorem (every complex commutative unital division Banach algebra is $\mathbb{C}$; and no, it's a different Mazur); the desired topology is the coarsest topology under which for each $f \in A$, the function $X \to \mathbb{C}$ sending x to $f(x) \in A/\mathfrak{m}_x \cong \mathbb{C}$ is continuous.

The space of valuations

Identify the set of maximal filters $\mathcal{M}(X)$ of an affinoid space $X = \operatorname{Max} A$ with maps $|\cdot|_a : A \to \mathbb{R}_{\geq 0}$ satisfying:

(i) $|fg|_a = |f|_a |g|_a$;

(ii) $|f+g|_a \leq \max\{|f|_a, |g|_a\}$;

(iii) $|c|_a = |c|$ for $c \in K$;

(iv) $|f|_a \leq \|f\|_X$ for $f \in A$.

We now topologize $\mathcal{M}(X)$ with the coarsest topology such that for each $F \in A$, the map $\mathcal{M}(X) \to \mathbb{R}_{\geq 0}$ sending $a \in \mathcal{M}(X)$ to $|f|_a$ is continuous; in other words, this is the topology induced by the product topology on $\mathbb{R}_{\geq 0}^A$. In particular, this space is Hausdorff and compact.

Aside: the Berkovich topology coincides not with the subspace topology on $\mathcal{P}(X)$, but for the quotient topology under a certain natural retraction $r : \mathcal{P}(X) \to \mathcal{M}(X)$ (see [FvdP, Definition 7.1.4]).

Note that this definition glues, so we can talk about $\mathcal{M}(X)$ even when X is not affinoid, although I'll refrain from doing so for the moment.

The point here is that $\mathcal{M}(X)$ with Berkovich's topology is "better connected" than X with its metric topology. Here's an easy case of this; more generally, Berkovich showed that "Smooth p-adic analytic spaces are locally contractible" (*Invent. Math.* **137** (1999), 1–84, though the proof is a lot harder.

Proposition 3. *Let A be a reduced affinoid algebra whose reduction is an integral domain, and put $X = \operatorname{Max} A$. Then $\mathcal{M}(X)$ is contractible.*

Proof. Let $i : \mathcal{M}(X) \to \mathcal{M}(X)$ be the identity map and let $j : \mathcal{M}(X) \to \mathcal{M}(X)$ be the map carrying everything to the "generic point" $|f|_a = \|f\|_{A,\text{spec}}$. (Note that I'm using here that the spectral seminorm is multiplicative; that's precisely what is guaranteed by the condition on the reduction of A.) Then there is an explicit homotopy $F(a,t) : \mathcal{M}(X) \times [0,1] \to \mathcal{M}(X)$ given by

$$F(a,t) = |\cdot|_a^t \|\cdot\|_{A,\text{spec}}^{1-t}.$$

□

More interesting things happen when you consider nonaffinoid spaces; for instance, Berkovich proved that if X is the analytification of an smooth irreducible projective curve over K of genus g having semistable reduction, then $\mathcal{M}(X)$ can be contracted to a closed subspace homeomorphic to the dual graph of the special fibre on a semistable model of X [FvdP, Theorem 7.2.4 for a more detailed statement]. I have no idea what $\mathcal{M}(X)$ looks like if X has non-semistable reduction, though I suppose you could probably figure it out by going up to an extension of K over which semistable reduction is acquired.

Example: the Berkovich closed unit disc

Let $X = \operatorname{Max} K\langle x\rangle$; let's identify $\mathcal{M}(X)$ explicitly in case K is algebraically closed. (I'll let you work out the general case for yourself.) Given an analytic point $a \in \mathcal{M}(X)$ and corresponding valuation $|\cdot|_a$, define the function $F_a : \mathfrak{o}_K \to [0,1]$ by $F_a(y) = |x-y|_a$. Such a function satisfies the triangle inequalities

$$|y-z| \leq \max\{F_a(y), F_a(z)\}, \qquad F_a(y) \leq \max\{F_a(z), |y-z|\}.$$

Let $\mathcal{N}$ be the set of functions F_a satisfying these inequalities, with the product topology (viewing functions as elements of the product of a bunch of copies of $[0,1]$ indexed by $\mathfrak{o}_K$). Then the map $\mathcal{M}(X) \to \mathcal{N}$ is a homeomorphism [FvdP, Lemma 7.2.1].

The functions F_a can be classified as follows.

- If $\inf(F_a) = 0$, then a is an ordinary point of X.
- If $\inf(F_a) > 0$ and F_a achieves its infimum at some $y \in \mathfrak{o}_K$, then a is a "generic point" of the disc $|x-y| \leq \inf(F_a)$ (which is affinoid if $\inf(F_a) \in |K^*|$), and one has $F_a(z) = \max\{|z-y|, \inf(F_a)\}$ for all $z \in \mathfrak{o}_K$ (so F_a coincides with the supremum norm on that small disc).
- If $\inf(F_a) > 0$ but F_a does not achieve its infimum, then K must fail to be spherically complete, and we can view F_a as the limit of the supremum norms on a decreasing sequence of discs. (This jibes with Jay's comment earlier that he had read somewhere that the Berkovich construction is somehow analogous to spherical completion.)

So what?

So far it looks like Berkovich's construction is a convenient gadget for visualization but doesn't suggest anything you couldn't do already. Not so! This theory turns out to be (and the Tate and Raynaud theories turn out not to be) just the thing for discussing the étale cohomology of rigid analytic spaces. The main point is that to do that, you need to have enough fibre functors for the étale topos (in more precise terms, you need a "conservative family of fibre functors"), and the analytic points give you just that. (Note that we've already run across this issue on the Zariski site.) That étale cohomology theory is vital for dealing with the sort of p-adically uniformized spaces occurring in Drinfeld's work on the Langlands correspondence for function fields.

There is a more topos-theoretic alternative if you prefer, which is Huber's theory of "adic spaces". However, in case you couldn't tell, I don't really go in for that sort of thing, and so I'm not going to discuss it further.

Exercises

1. (from [FvdP, Exercise 7.2.5]) Suppose K has characteristic $p > 0$. Put $X = \operatorname{Max} K\langle x\rangle$. Prove that the map $X \to X$ induced by $x \mapsto x^p - x$ is finite and unramified at each ordinary point of X, but ramifies at one analytic point of X