18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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More on coherent sheaves

References: [FvdP, Chapter 4]. Kiehl's original papers (in German) are: Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedische Funktionentheorie, *Inv. Math.* **2** (1967), 191–214; and Theorem A und B in der nichtarchimedische Funktionentheorie, *Inv. Math.* **2** (1967), 256–273.

Addendum: comment about generic fibres

I forgot to mention in the previous handout that the generic fibre construction loses information in a very precise sense. If P is a formal scheme of finite type over $\mathfrak{o}_K$, and P' is a blowup of $\mathfrak{o}_K$ within the special fibre, then the induced map from the generic fibre of P' to that of P is an isomorphism.

Example: if $P = \operatorname{Spf} \mathfrak{o}_K\langle x, y\rangle$, and P' is the blowup along the ideal (x, y, π) for some $\pi \in \mathfrak{m}_K$, then in both cases the generic fibre consists of the closed points of the affine plane over K represented by geometric points (a, b) with $|a| \leq 1$ and $|b| \leq 1$.

In fact, Raynaud proved an equivalence of categories between a certain category of rigid spaces, and a certain category of formal schemes localized at the blowups in the special fibre. (I don't know the precise formulation offhand; will go look it up later.) This cuts both ways: sometimes it's easier to work with rigid spaces than formal schemes because you would just as soon ignore the blowups in the special fibre. But sometimes you can't make a construction in the rigid setting without keeping track of some choice of an associated formal scheme (as in Laurent Fargues's seminar talk a couple of weeks ago).

Coherent sheaves on affinoid spaces

Last time I defined a coherent sheaf on a rigid space to be a sheaf which, locally on some admissible affinoid covering, is isomorphic to the sheaf generated by a finitely generated module over the corresponding affinoid algebra. The Gerritzen-Grauert-Tate theorem implies that you actually get a sheaf this way, but it doesn't imply that this sheaf has "enough" sections. This is fixed by the following theorem of Kiehl, whose proof I'll sketch here; see [FvdP, Section 4.5] for details.

Theorem 1 (Kiehl). *Let $\mathcal{F}$ be a coherent sheaf on an affinoid space $X = \operatorname{Max}(A)$, and put $M = \mathcal{F}(X)$. Then M is finitely generated, and $\mathcal{F}$ is isomorphic to the coherent sheaf associated to M.*

Sketch of proof. As in the Gerritzen-Grauert-Tate argument, one reduces to the case of a Laurent covering given by a single f. (See [FvdP, Section 4.5] for more on this reduction.) That puts us in the following situation. We are given $f \in A$, finitely generated modules M_+ and M_- over $A\langle f\rangle$ and $A\langle f^{-1}\rangle$, respectively, and an isomorphism between $M_+ \otimes A\langle f, f^{-1}\rangle$

and $M_- \otimes A\langle f, f^{-1}\rangle$. For $M = \mathcal{F}(X)$, we must show that the maps $M \otimes A\langle f\rangle \to M_+$ and $M \otimes A\langle f^{-1}\rangle \to M_-$ are surjective.

One can easily reduce this to just checking for M_- (as in [FvdP, Lemma 4.5.5], but they seem to get it backwards; see below). For this part, one needs a form of Cartan's lemma: there exists $c > 0$ such that any invertible $n \times n$ matrix U over $A\langle f, f^{-1}\rangle$ with $\|U - I_n\| < c$ (where the matrix norm is the maximum over entries) factors as U_+U_-, with U_+ invertible over $A\langle f\rangle$ and B invertible over $A\langle f^{-1}\rangle$. See [FvdP, Lemma 4.5.3] for this calculation. (The basic idea is to split U *additively* as $I + V_+ + V_-$, where V_+ has only nonnegative powers of V, V_- has only nonnegative powers of V_-, and both V_+ and V_- have small norm. Then you replace U by $(1 - V_+)U(1 - V_-)$ and repeat; if c is small enough, this process converges to the desired factorization.)

To check surjectivity of $M \otimes A\langle f\rangle \to M_+$ now, you choose a set of n generators of M_+ and a set of n generators of M_- (for some n), and write down change-of-basis matrices U and V between the two sets of generators over $A\langle f, f^{-1}\rangle$. Since the image of A in $A\langle f\rangle$ is dense, we can approximate U closely by a matrix over A, so that $\|(U' - U)V\| < c$ with c as above. We can then factor $I_n - (U' - U)V = U_+U_-$ as above, and changing basis from M_+ via U_+ gives us a set of generators which are defined on both subspaces, yielding the surjectivity.

Given the surjectivity, we can choose a finitely generated submodule M_1 of M which generates both M_+ and M_-. Let $\mathcal{M}_1$ be the associated sheaf: then $\mathcal{M}_1 \to \mathcal{F}$ is surjective. Let $\mathcal{G}$ be the kernel of that map; then $\mathcal{G}$ is also coherent and given by finitely generated modules on the two pieces of the cover, so we can find a surjection $\mathcal{M}_2 \to \mathcal{G}$, where $\mathcal{M}_2$ is the coherent sheaf associated to the finitely generated module M_2. Hence $\mathcal{F}$ is the cokernel of the map $\mathcal{M}_2 \to \mathcal{M}_1$; by acyclicity, its global sections are precisely M_2/M_1, and the associated sheaf is precisely $\mathcal{F}$ because they match up on the two opens. □

This means that we can glue coherent sheaves on any admissible cover of X (I may have said this earlier without justification).

Example: the sheaf of differentials

An important example of a coherent sheaf is the sheaf of continuous differentials.

Theorem 2. *Let A be an affinoid algebra. There exists a finitely generated A-module $\Omega_{A/K}$ equipped with a K-linear derivation $d : A \to \Omega_{A/K}$ with the following universal property: given any finitely generated A-module M equipped with a K-linear derivation $D : A \to M$, there exists a unique A-module homomorphism $l : \Omega_{A/K} \to M$ with $D = l \circ d$.*

Proof. In case $A = T_n = K\langle x_1, \dots, x_n\rangle$, this is easy to check: take $\Omega_{A/K} = Adx_1 + \cdots + Adx_n$ together with the formal total differential. That is,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i,$$

where the partial derivation is done formally on the series.

For general A, pick a presentation $A \cong T_n/(f_1, \dots, f_m)$, and put

$$\Omega_{A/K} = \Omega_{T_n/K}/(T_n df_1 + \cdots + T_n df_m),$$

with the induced derivation from T_n. □

The module $\Omega_{A/K}$ is called the *module of finite differentials*, or the *universal finite differential module*. It is obviously unique up to unique isomorphism, so one gets from it a coherent sheaf of differentials Ω_X on any rigid space X.

Note that the module of finite differentials is *not* always the same as the module of Kähler differentials, because the latter is sometimes badly behaved. In fact, it can be shown (though I don't remember a reference offhand) that the module of finite differentials is the maximal separated quotient of the completion of the ordinary module of Kähler differentials; it is thus sometimes called the *module of continuous differentials*, because the latter description makes it universal for continuous derivations into any Banach A-algebra.

Here are some fun facts about modules of differentials. (See [FvdP, Theorem 3.6.3], although they mostly just refer to Springer LNM 38 "Differentialrechnung in der analytische Geometrie"):

Proposition 3. *Let A be an affinoid algebra which is an integral domain, and let F be its fraction field.*

(a) The dimension of $\Omega_{A/K} \otimes_A F$ over F equals the Krull dimension d of A.

(b) Let $\mathfrak{m}$ be a maximal ideal of A. Then the following are equivalent.

(i) A is smooth over K at $\mathfrak{m}$. (That means that locally near $\mathfrak{m}$, A can be written as the vanishing locus in some affine space $\operatorname{Max} T^n$ *of some number m of functions whose $m \times n$ matrix of partial derivatives has maximal rank. If K is perfect, this is equivalent to A being regular at $\mathfrak{m}$, i.e., the localization $A_\mathfrak{m}$ being a regular local ring.)*

(ii) The localization $A_\mathfrak{m} \otimes_A \Omega^{A/K}$ is free over $A_\mathfrak{m}$ (of rank d).

(iii) The dimension of $\Omega_{A/K}/\mathfrak{m}\Omega_{A/K}$ over $A/\mathfrak{m}$ is d.

Closed analytic subspaces

Let X be a rigid space over k, and let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}$. Then we get a subspace of X associated to $\mathcal{I}$ whose points are the support of the coherent sheaf $\mathcal{O}/\mathcal{I}$ (with topology induced from X), and whose structure sheaf is the restriction of $\mathcal{O}/\mathcal{I}$. On an affinoid space, this of course corresponds to viewing the vanishing locus of some ideal of functions as the Max of the quotient affinoid algebra. Any such subspace of X is called a *closed analytic subspace.*

Next time: GAGA

Next time, I'll talk about separatedness and properness for rigid spaces and sketch the proof of Kiehl's rigid GAGA theorem.

Exercises

1. Assume that K has characteristic zero. Let X be a *smooth* rigid space, let $\mathcal{F}$ be a coherent sheaf on X, and suppose that there exists a K-linear connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/K}$. That is, given a section s of $\mathcal{F}$ and a function f, we have $\nabla(sf) = f\nabla(s) + s \otimes df$. Prove that $\mathcal{F}$ must be locally free. (Hint: replace the local ring at a maximal ideal by its completion and check there that the torsion submodule must vanish.)