**18.727, Topics in Algebraic Geometry (rigid analytic geometry)**
**Kiran S. Kedlaya, fall 2004**
**$p$-adic cohomology, part 1**

**References:** [FvdP, Chapter 7]. Also, Berthelot's "Géométrie rigide et cohomologie des variétés algébriques de caractéristique $p$" is a good (if cursory) overview; more on his other papers below.

But first, I have a bunch of corrections and addenda from the discussion of Berkovich's construction.

## Corrections from last time

As noted in class, the map $\sigma : X \to \mathcal{P}(X)$ for a $G$-topological space $X$ is not actually a continuous morphism of sites. Nonetheless, it "gives rise to" a "direct image" $(\sigma_* \mathcal{F})(U) = F(\mathcal{P}(U))$ and "inverse image" $\sigma^* \mathcal{F}$ the sheafification of the presheaf $U \mapsto \lim_{\to} \mathcal{F}(V)$, where $V$ runs over admissible sets with $U \subseteq \mathcal{P}(V)$. See [FvdP, Theorem 7.1.2]; note that it requires the $G$-topological space $X$ to be quasi-compact. The point is that $\sigma_*$ is an equivalence between the category of abelian sheaves on $\mathcal{P}(X)$ and on $X$, with quasi-inverse $\sigma^*$. [FvdP] punts on the question of whether the same is true for sheaves of *sets*, which would say that these two maps give an isomorphism of topoi between $X$ and $\mathcal{P}(X)$. (I had thought this followed formally by applying the "free abelian group generated by" functor; a clarification would be appreciated.)

I mentioned in class there is a retraction map $r : \mathcal{P}(X) \to \mathcal{M}(X)$ when $X$ is a rigid space. Here's one way to define this map: if $p$ is a prime filter, $r(p)$ is the unique maximal filter containing $r$. The uniqueness is a theorem, but not a hard one: if $p$ corresponds to a valuation of some rank, $m$ corresponds to the valuation where you "ignore all terms not of highest order". For instance, if $p$ corresponded to a valuation into $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ ordered lexicographically, then $m$ would correspond to the projection of that valuation onto the *first* copy of $\mathbb{R}$. For another approach, see [FvdP, Definition 7.1.4].

My alleged proof of the contractibility of the Berkovich space for an affinoid algebra whose reduction is an integral domain doesn't work, and I don't see how to fix it offhand. In fact, I'm not even sure if the result as stated is true! On the other hand, it's definitely true that the Berkovich space of $K\langle x \rangle$ is true, because from the generic point of any disc you can travel to the generic point of the disc with the same center and any radius, including 1, which is the generic point of the whole space. (Compare [FvdP, Lemma 7.2.2].) I think something similar works for $K\langle x_1, \dots, x_n \rangle$, but I didn't check details.

## Addendum: Globalizing Berkovich

The discussion from last time about $\mathcal{M}(X)$ for $X$ an affinoid space hides some of the subtle distinctions between our category of rigid spaces and Berkovich's category of "analytic spaces". Indeed, Berkovich changed his category between his original monograph and his

IHES paper; in any case, his category is somewhat richer than the one we have in mind. (See his IHES paper for details.)

For Berkovich, an "affinoid algebra" is anything covered by the ring of functions on the closed polydisc $|x_1| \leq r_1, \ldots, |x_n| \leq r_n$ for *any* $r_1, \ldots, r_n > 0$; in particular, they don't have to be in the divisible closure of $|K^*|$. Our affinoid algebras are "strictly affinoid" in his terminology. (Berkovich also allows $K$ to carry the trivial valuation, to make certain arguments more uniform.)

A *quasi-net* on a topological space $X$ is a collection $\tau$ of subsets of $X$ with the property that for any $x \in X$, we can find $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \cdots \cap V_n$ and the set $V_1 \cup \cdots \cup V_n$ is a neighborhood of $x$. (Note that I didn't say "open neighborhood". I believe the intent is to allow any set containing an open subset containing $x$.) A *net* is a quasi-net whose restriction to $U \cap V$, for any $U, V$ in the net, is again a quasi-net. Example: any basis of open sets. More amusing example: divide the plane into unit squares like an infinite chessboard, and take the closed unit squares in this decomposition. (Warning: this is Berkovich's terminology, but is inconsistent with the preexisting term "net" in ordinary topology, which refers to a generalized sequence!)

An *affinoid atlas* on a locally Hausdorff topological space $X$, equipped with a net $\tau$ of compact subsets, is a homeomorphism of each $U \in \tau$ with an affinoid space, such that for $U, V \in \tau$ with $U \subseteq V$, $U$ is identified with an affinoid subspace of $V$. An *analytic space* is a space equipped with an affinoid atlas, modulo inverting "quasi-isomorphisms". (Or if you prefer, you can insist that the affinoid atlas be maximal, and then there's nothing to invert. Aside: I think you can give a ringed space definition instead, but I don't know any such definition in the literature.)

The natural subspaces of an analytic space $X$ are *analytic domains*. An analytic domain is a subset $Y$ of $X$ such that for any $y \in Y$, you can find affinoid subspaces $U_1, \ldots, U_n$ in $Y$ with $y \in U_1 \cup \cdots \cup U_n$ and $U_1 \cup \cdots \cup U_n$ being a neighborhood.

Relationship with our other construction: the natural functor from our rigid spaces to Berkovich spaces is fully faithful, and it induces an equivalence between the category of paracompact strictly analytic spaces in Berkovich's sense, and the category of quasi-separated rigid spaces admitting an admissible affinoid covering of finite type. (Of course, most spaces you will meet in practice are of this form...)

## More addenda

Brian Conrad reminds me that there is an unpublished German PhD thesis (by Ulrike Kopf) that works out a lot of the details of rigid GAGA. I have a copy in my office somewhere. (I think Brian's original source for this was Richard Taylor.)

Brian also points out that Čech and sheaf cohomology agree for any sheaf on any quasi-separated rigid space (reminder: that means the diagonal $\Delta_X : X \to X \times X$ is a locally closed immersion), though this isn't in the literature. He sent me a handout (from a course he taught) with the details; I may incorporate these into a handout at some point.

Clarification: Temkin's work postdates the paper by Conrad that I mentioned last time, so you have to look there directly: "On local properties on non-Archimedean analytic spaces

I, II". One of Temkin's key results is that every point of a Berkovich space admits an affinoid neighborhood. (This is true by design for the Berkovichifications of rigid analytic spaces, but it's not so obvious in general.) Aside: Berkovich's definition of "proper" is purely topological (see his ICM talk); it looks formally stronger than Kiehl's definition when applied to a rigid space, and maybe it even is stronger. In any case, it satisfies Kiehl's finiteness theorem, so maybe it's really the correct definition...

## Motivation: algebraic de Rham cohomology

Now, on to today's topic.

Grothendieck's letter to Atiyah (published in IHES as "On the de Rham cohomology of algebraic varieties") explains the construction of algebraic de Rham cohomology on a smooth variety $X$ over a field $K$ of characteristic zero; basically, it's the hypercohomology of the complex of algebraic differentials $0 \to \Omega^0_{X/K} \to \Omega^1_{X/K} \to \cdots$. Over $K$ with $X$ proper, this agrees with holomorphic de Rham cohomology by GAGA, which agrees with smooth de Rham cohomology by the Dolbeaut lemma (see Griffiths-Harris), which agrees with topological cohomology with $\mathbb{C}$-coefficients by de Rham's theorem.

Hartshorne (published in IHES with the same title!) worked out a construction that works for nonsmooth varieties also. (There is also Grothendieck's related but more coordinate-free approach via the infinitesimal site, i.e., via "crystals", which I'm going to ignore for now.) The basic idea is to replace $X$ Zariski (or étale) locally by its formal completion in some smooth ambient variety, and check that the resulting de Rham complex is "canonically independent of the choice of compactification".

The idea behind Berthelot's rigid cohomology (which was inspired by Dwork's proof of rationality of the zeta function as well as by ideas of Grothendieck and Monsky-Washnitzer) is to use a similar paradigm to construct de Rham cohomology of varieties in characteristic $p > 0$, by first passing to a $p$-adic lifting. Loosely speaking, the choice of the lift should drop out "at the level of homotopy".

## Dagger algebras and de Rham cohomology

Additional references: besides [FvdP, 7.5], see also the papers of Elmar Grosse-Klönne (but make sure also to read the MathSciNet review of "Rigid analytic spaces with overconvergent structure sheaf" for an important erratum).

The *Monsky-Washnitzer algebra* $K\langle x_1, \ldots, x_n\rangle^\dagger$ is the set of formal power series $\sum c_I x^I$ which converge on some polydisc $|x_i| \leq c$ for some $c > 1$ (but the choice of $c$ depends on the particular series). This is a dense subring of the Tate algebra $K\langle x_1, \ldots, x_n\rangle$, and you can topologize it that way (I'll call that the "affinoid topology"). On the other hand, it's sometimes better to topologize it as as the direct limit of the coordinate rings of the closed polydiscs of all radii $> 1$ (I'll call that the "dagger topology").

One can verify that pretty much all of the standard facts about Tate algebras apply to these guys too. For instance, they satisfy Weierstrass preparation [FvdP, Lemma 7.5.1], Noether normalization, et cetera. The one thing they do which Tate algebras don't is admit a Poincaré lemma: the partial derivative maps $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ on $K\langle x_1, \ldots, x_n\rangle$ are all surjective!

A *dagger algebra* (or *overconvergent affinoid algebra*) is a quotient of a $K\langle x_1, \ldots, x_n\rangle^\dagger$; it turns out any map between these is compatible with both the induced affinoid and dagger topologies. The Max of such an algebra is the same as that of its (affinoid) completion. There is an analogous notion of *dagger subspace* of $\operatorname{Max} A$ for $A$ a dagger algebra; any rational subspace is a dagger subspace, and any dagger subspace is an affinoid subspace, but I don't know whether every affinoid subspace is a dagger subspace. (Neither does [FvdP], and as far as I can tell neither does Grosse-Klönne.) But never mind that; you still get the same strong $G$-topology on $\operatorname{Max} A$ using just the dagger subdomains (by Gerritzen-Grauert), and so you get a subsheaf $\mathcal{O}^\dagger$ of $\mathcal{O}$, the *overconvergent structure sheaf.*

A *dagger space* is a locally $G$-ringed space which locally on some admissible cover looks like an affinoid space equipped with an overconvergent structure sheaf. (Note: the forgetful functor from dagger to rigid spaces is faithful but not fully faithful; that is, there are different incompatible dagger structures on a typical affinoid algebra.) Besides the structure sheaf, every dagger space also carries a sheaf of (overconvergent) continuous differentials $\Omega^\dagger_X$, whose definition I'll leave to your imagination. The *de Rham complex* on $X$ is of course

$$\cdots \to \Omega^{i,\dagger}_X = \wedge^i_{\mathcal{O}^\dagger_X} \Omega^\dagger_X \to \cdots$$

and its hypercohomology is the *de Rham cohomology* of $X$. (If you don't like hypercohomology, think for now about an overconvergent affinoid space, in which case all the sheaf cohomology drops out and it's just the cohomology of the ordinary complex of global sections.)

It turns out [FvdP, Proposition 7.5.13] that the de Rham cohomology of a dagger space actually depends only on its underlying rigid space; moreover, they are functorial in the rigid spaces. (Idea: say $f : X \to Y$ is a map between rigid spaces, and that each of $X$ and $Y$ has been equipped with a dagger structure. By a certain approximation argument, you can closely approximate $f$ by an actual map between the dagger spaces. You then show that there is a homotopy between the maps on de Rham complexes induced by any two choices of $f$.)

## Monsky-Washnitzer cohomology

I promised earlier that this would have something to do with characteristic $p$, and indeed it does!

Let $K$ be a complete discretely valued field of characteristic 0, whose residue field $k$ has characteristic $p > 0$. Let $X = \operatorname{Spec} \overline{A}$ be a smooth affine scheme over $k$.

By a theorem of Elkik (for a more modern treatment, see the paper by Arabia: "Relèvements des algèbres lisses et des leurs morphismes"), you can lift $\overline{A}$ to a smooth $\mathfrak{o}_K$-algebra $A$. The *Monsky-Washnitzer cohomology* of $X$ is the overconvergent de Rham cohomology of the affinoid space $\operatorname{Max} \widehat{A}$; by the same arguments as in the previous section, it is independent of the choice of the lift and functorial in $X$. It is finite dimensional over $K$, but this is not so easy to prove: it was first shown by Berthelot using rigid cohomology (see next handout).

Monsky and Washnitzer, inspired by Dwork's proof of rationality of the zeta function, gave a Lefschetz trace formula for Frobenius for their cohomology: if $k = \mathbb{F}_q$ and $X$ is of pure dimension $d$, then

$$\#X(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{Trace}((q^d F_*^{-1})^n, H^i_{MW}(X)),$$

where $F_*$ denotes the map induced by the algebraic Frobenius $t \mapsto t^q$. See Monsky-Washnitzer's "Formal Cohomology I" and Monsky's "Formal Cohomology II, III" (or [FvdP, 7.6]). The basic idea here is to establish enough topological formalism so that you can do excision to reduce to the case where $X$ has *no* $\mathbb{F}_{q^n}$-rational points, and then show that the right side is forced to vanish. In fact, it already vanishes *on the chain level*!

Note: making the previous paragraph makes some sense requires a bit of $p$-adic functional analysis. Even if you know finite dimensionality in cohomology (which Monsky and Washnitzer didn't), the de Rham complex consists of infinite dimensional vector spaces over $K$. So you need to work with nuclear operators; in particular, you have to choose a so-called "Dwork operator" (essentially a one-sided inverse of Frobenius), show that it is nuclear, and then check (by a neat calculation) that its trace is forced to vanish.

It cannot go unsaid here that Monsky-Washnitzer cohomology is a fabulous tool for computing zeta functions and other cohomological invariants of varieties over finite fields, particularly when they lift nicely to characteristic zero. For instance, "Kedlaya's algorithm" computes the zeta function of a hyperelliptic curve over a finite field (of not-too-large characteristic) by computing on a dagger algebra and being careful about $p$-adic precision; see [FvdP, 7.6] or my paper "Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology".

## Next time: rigid cohomology

Next time, I'll tack on a short discussion of how to globalize the ideas of the previous section, and how to extend them to singular varieties. This will give Berthelot's rigid cohomology.