18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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p-adic cohomology, part 2

References: this stuff is not written down all in one place. The survey article by Berthelot ("Geometrie rigide et cohomologie...") I mentioned last time is a good start. For more details, see his Inventiones article "Finitude et purité cohomologique en cohomologie rigide". Further references appear within the text.

Rigid cohomology

Last time, I described a good de Rham-type cohomology theory for smooth affine varieties over a field k of characteristic $p > 0$, based on taking liftings up to characteristic zero. You can "sheafify" that theory to get a good theory for general smooth varieties. (Be careful: the de Rham complexes were not functorial, only functorial "up to homotopy".)

What about general varieties? As I mentioned last time, for algebraic de Rham cohomology, a good approach is to locally embed a general variety into a smooth variety and work on the formal completion there. That "smudges out" the singularity and gives you a smooth space with the same "homotopy type". The idea here is similar, but the formal completion gets replaced by something a bit bigger.

Let X be a variety over a field k of characteristic $p > 0$. Suppose I have an open immersion $X \hookrightarrow Y$, where Y is another k-variety, and a closed immersion $Y \hookrightarrow P_k$, where P is a smooth formal scheme of finite type over $\mathfrak{o}_K$ (and P_k is its special fibre). A good example to visualize is $P = \mathbb{P}^n$. Let P_K denote the rigid analytic generic fibre of P; remember that this has a specialization map $\mathrm{sp} : P_K \to P_k$. For $S \subseteq P_k$, define the *tube* $]S[= \mathrm{sp}^{-1}(S)$. I'll write it as $]S[_P$ if I need to specify P, which will happen a bit later.

The most precise analogue of the formal completion construction would be to consider the tube $]X[$, but that has the same sorts of problems as we saw last time with the de Rham cohomology of a closed disc. Instead, we must take something slightly bigger.

Reminder: for U an admissible subset of a G-topological space X such that $X \setminus U$ is also admissible, we say V is a *strict neighborhood of U in X* if V is an admissible subset containing U, and $\{V, X \setminus U\}$ form an admissible cover of X.

So what we really want to look at is not the tube $]X[$, but a strict neighborhood of $]X[$ in $]Y[$. If P happens to be affine, this can be described more concretely: if $f_1, \dots, f_n \in \Gamma(\mathcal{O}, P)$ cut out Y within P_k, we can describe $]Y[$ as the locus where $|f_1|, \cdots, |f_n| < 1$, and in particular we can make an admissible cover out of the sets U_ϵ where $|f_1|, \dots, |f_n| \leq \epsilon$ for $0 < \epsilon < 1$. (Strictly speaking, I should be taking ϵ in the divisible closure of K^*, but to keep notation simple, I'm going to ignore that technicality consistently.) If $g_1, \dots, g_m \in \Gamma(\mathcal{O}, P)$ cut out $X \setminus Y$ within X, we can characterize strict neighborhoods of $]X[$ in $]Y[$ as follows: an admissible open V in $]Y[$ containing $]X[$ is a strict neighborhood if and only if its intersection with the affinoid U_ϵ contains the set on which

$$|f_1|, \cdots, |f_n| \leq \epsilon, \qquad |g_1|, \cdots, |g_m| \geq \delta$$

for some $\delta \in (0,1)$. (If δ were 1, we would just be picking up $]X[\cap U_\epsilon$.)

We get an exact functor $j^\dagger$ on abelian sheaves E on $]Y[$ by the formula

$$j^\dagger E = \varinjlim j_{V*} j_V^{-1} E,$$

the limit taken over strict neighborhoods V of $]X[$ in $]Y[$. This is sometimes called the "overconvergent sections functor"; we are particularly interested in applying it to the de Rham complex.

Example: if $X = \mathbb{A}^n$, $Y = \mathbb{P}^n_k$, and $P = \mathbb{P}^n_{\mathfrak{o}_K}$, then $]X[= \operatorname{Max} K\langle x_1, \dots, x_n\rangle$, and $\Gamma(j^\dagger \mathcal{O},]X[) = K\langle x_1, \dots, x_n\rangle^\dagger$.

Suppose you can set this situation up with P proper over $\mathfrak{o}_K$. Then we want to define the *rigid cohomology* of X with coefficients in K as the hypercohomology

$$H^i_{\mathrm{rig}}(X/K) = \mathbb{H}^i(]Y[_P, j^\dagger \Omega_{]Y[}).$$

However, one has to check that this is independent of the choice of P, as well as functorial in X. The basic idea is this: if $u : P' \to P$ is proper, you want to compare the rigid cohomology of X computed within P and within P'. (If you can do that, you can compare two choices of P by comparing each to the fibre product over $\mathfrak{o}_K$.) That you do by showing that the canonical morphism

$$j^\dagger \Omega_{]Y[_P} \to \mathbb{R}u_{K*} j^\dagger \Omega_{]Y[_{P'}}$$

is an isomorphism (e.g., see Théorème 1.4 of Berthelot's Inventiones paper). This in turn follows from the "strong fibration theorem", which says that locally on P_K, a sufficiently small strict neighborhood of $]X[_{P'}$ in $]Y[_{P'}$ looks like the product of an open unit polydisc with a strict neighborhood of $]X[_P$ in $]Y[_P$. Note how crucial it is here that the Poincaré lemma holds on an *open* polydisc! (To get functoriality along $X \to X'$, embed X into P and X' into P', then embed the graph of the map into $P \times P'$, etc.)

This is enough to define rigid cohomology of quasi-projective X. For general X, you need to cover X with, say, affines and use an appropriate Čech complex to define X. Berthelot never bothers to explain this rigorously; probably the right way to say this is in the language of simplicial sets; this is the way Shiho proceeds in his two papers "Crystalline fundamental groups..., I, II".

You won't be surprised to know that there is a comparison theorem between this construction and Monsky-Washnitzer cohomology (Théorème 1.10 in Berthelot's Inventiones paper). There is also a comparison between rigid and crystalline cohomology *after* tensoring the latter up to K (Théorème 1.9). But the latter is really an integral cohomology theory (defined over the Witt ring $W(k)$), so the passage to cohomology really loses some information (e.g., about the failure of the Hodge-de Rham spectral sequence to degenerate). Rigid cohomology does seem to be a "universal" p-adic cohomology with field coefficients, or if you like, a universal p-adic Weil cohomology.

There is also a construction of rigid cohomology with supports in a closed subscheme, and of cohomology with compact supports. These are needed to talk about excision sequences and Poincaré duality, and to correctly formulate the Lefschetz trace formula for Frobenius (which works on cohomology with supports, and doesn't require properness). See Berthelot for more details.

Isocrystals

How do you put coefficients into this theory? In de Rham cohomology, you use local systems: vector bundles equipped with an integrable connection. The point is that you need the connection in order to have complex maps on the vector bundle tensored with the original de Rham complex.

In rigid cohomology, you do something similar; this is described best in an unpublished preprint of Berthelot called "Cohomologie rigide, I", available on his web site at Rennes. (He has some other useful papers there, but the others are all published somewhere.) The basic idea goes back to Grothendieck's algebraic interpretation of integrable connections: given a vector bundle E on a space X, an integrable connection should come from a "parallel transport" isomorphism $\pi_1^* E \to \pi_2^* E$ on the formal completion along the diagonal $\Delta \subseteq X \times X$ by taking first-order infinitesimals. (On higher order infinitesimals, this isomorphism looks in coordinates like it's being computed by Taylor series.)

In my notation from before, what an "overconvergent isocrystal" on X should be is a vector bundle $\mathcal{E}$ on some unspecified strict neighborhood of $]X[_P$ in $]Y[_P$ plus a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1$, which induces a parallel transport isomorphism on some strict neighborhood of $]X[_{P'}$ in $]Y[_{P'}$, where $P' = P \times_{\mathfrak{o}_K} P$. This turns out to be the right construction; this category is independent of the choice of P (in that if P' is another choice, the pullback functor along $P' \to P$ is an equivalence). It's called the category of *isocrystals on X (over K) overconvergent along Z*, where $Z = Y \setminus X$. If Y is proper, these are just *overconvergent isocrystals on X*. (There are also *convergent isocrystals on X*, which are defined just on $]X[$ itself, but you still have to make the nontrivial restriction that the parallel transport isomorphism is defined on all of $]X[_{P \times P}$.)

These are known to have reasonable cohomological properties under some additional hypotheses: K should be discretely valued and the isocrystals should carry "Frobenius structure", i.e., an isomorphism between the isocrystal and its pullback along some lift of the absolute Frobenius. See my preprint "Finiteness of rigid cohomology with coefficients". You can also scrape by just enough formalism mirroring that of étale cohomology to prove the Weil conjectures, by reproducing Laumon's simplified proof of Deligne's "Weil II" theorem (essentially, the Weil conjectures with coefficients); see my preprint "Fourier transforms and p-adic 'Weil II"'.

A big problem right now is to enlarge this category of coefficients; the category of isocrystals has no hope of being stable under direct images, because objects are of constant rank. (In étale cohomology, this would be like working only with the lisse (smooth) sheaves and not the constructible ones.) The fix in algebraic de Rham theory is to work with $\mathcal{D}$-modules, for $\mathcal{D}$ a suitable sheaf of differential operators. (You can imagine vector bundles with integrable connection as carrying an action of certain differential operators via the connection.) There you have a good finiteness notion called "holonomicity"; reproducing that in some sort of p-adic $\mathcal{D}$-module context is a problem Berthelot has been stuck on for a long time. (The $\mathcal{D}$-modules in question manifest already in the Weil II argument mentioned above, but in a rather simple way that doesn't cause much trouble.)

Upshot: although rigid cohomology comes a long way towards the dream of finding a way to interpret Dwork's work on zeta functions in terms of a fully functional p-adic cohomology theory for varieties over a field of positive characteristic, there are a lot of technical issues that remain unresolved.