18.727, Topics in Algebraic Geometry (rigid analytic geometry)
Kiran S. Kedlaya, fall 2004
More on G-topologies, part 1 (of 2)

Leftover from last time: separating discs

Here's a more precise answer to Andre's question about separating discs (which you need in order to do the reduction of the theorem identifying A_F to the connected case). I'll leave it to you to work out the (easy) reduction of the general case to this specific case.

Proposition 1. *Given $r_1 < r_2$ and $c > 0$, there exists a rational function $f \in K(x)$ such that*

$$\sup\{|f(x) - 1| : |x| \leq r_1\} \leq c, \qquad \sup\{|f(x)| : |x| \geq r_2\} \leq c.$$

Proof. For simplicity, I'm going to consider the special case where there exists $r \in |K^*|$ with $r_1 < r < r_2$, and leave the general case as an exercise. Choose $a \in K^*$ with $|a| = r$, and put $g(x) = a/(a - x)$. Then for $|x| \leq r_1$,

$$|g(x) - 1| = |x/(a - x)| = |x|/r_1 < 1$$

while for $|x| \geq r_2$,

$$|g(x)| = |a/(a - x)| = r/|x| < 1.$$

Now take N large enough that $(r/r_2)^N \leq c$. Then g^N has the desired bound on the outer disc; in fact, so does any polynomial in g^N with integer coefficients. On the other hand, we also have that $|g^N - 1| \leq r_1/r < 1$ on the inner disc; so we can take

$$f = (1 - g^N)^M - 1$$

for M so large that $(r_1/r)^M \leq c$. □

G-topologies and Grothendieck topologies

The notion of a G-topology is a special case of the concept of a Grothendieck topology. Given a category $\mathcal{C}$ admitting finite products, a *Grothendieck topology* on $\mathcal{C}$ consists of, for each $X \in \mathrm{Obj}(\mathcal{C})$, a family $\mathrm{Cov}(X)$ of "coverings" of X, where a covering is a set of arrows $\{U_i \to X\}_{i \in I}$ in $\mathcal{C}$. (Note that I am wantonly ignoring foundational set-theoretic issues; for instance, $\mathrm{Cov}(X)$ is typically a proper class, not a set.) The coverings must then satisfy the following properties.

(a) For each $X \in \mathrm{Obj}(\mathcal{C})$, $\{X \to X\} \in \mathrm{Cov}(X)$ (trivial coverings always exist).

(b) If $Y \to X$ is an arrow in $\mathcal{C}$ and $\{U_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$, then $\{U_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$ (coverings pull back).

(c) If $\{U_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$, and for each $i \in I$, $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(U_i)$, then $\{V_{ij} \to U_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$ (coverings can be composed).

A G-topology is then just a Grothendieck topology on a category of subsets of a set in which morphisms are inclusions, the empty set and the whole set both appear, and in which finite products (i.e., finite intersections) exist.

Consequence: whatever you know about Grothendieck topologies will be true here, and is not really any easier to prove here than in general. For instance, the category of sheaves of abelian groups has enough injectives.

Sheaf cohomology and Čech cohomology

Say I have a presheaf $\mathcal{F}$ on a space X equipped with a G-topology, and an admissible covering $\{U_i\}_{i \in I}$ of X. Then one can make a Čech complex in the usual fashion, as follows. Let C^n be the product of $\mathcal{F}(U_{i_0} \cap \cdots U_{i_n})$ for all $(n+1)$-tuples $(i_0, \dots, i_n) \in I^{n+1}$. Then one can make a map $d^n : C^n \to C^{n+1}$ such that given an element $\xi = (\xi(i_0, \dots, i_n))$, $d(\xi)$ has its $(i_0, \dots, i_{n+1})$ component equal to

$$\sum_{j=0}^{n+1} (-1)^j \xi(i_0, \dots, \widehat{i_j}, \dots, i_{n+1}),$$

where the hat means you omit that term.

As usual, the maps d^n satisfy $d^{n+1} \circ d^n = 0$, so you get a complex $0 \to C_0 \to C_1 \to \cdots$. We write $\check{H}^n(\mathcal{F}, \{U_i\})$ for the cohomology of this complex, and call it the *Čech cohomology for the sheaf $\mathcal{F}$ and the covering* $\{U_i\}$. Note that the only 1-element covering is $\{X\}$, so $C_0 = \mathcal{F}(X)$, and you have a natural map $\mathcal{F}(X) \to \check{H}^0(\mathcal{F}, \{U_i\})$; the presheaf $\mathcal{F}$ is a sheaf if and only if this map is a bijection for any covering $\{U_i\}$.

Again as usual, if $\{V_j\}$ is a refinement of $\{U_i\}$ (in the sense that each V_j is contained in some U_i), you get natural maps $\check{H}^n(\mathcal{F}, \{U_i\}) \to \check{H}^n(\mathcal{F}, \{V_j\})$. The direct limit of these is called $\check{H}^n(\mathcal{F})$, or $\check{H}^n(X, \mathcal{F})$ if you want to remind yourself which space you are working on. Leray's theorem applies in this context: if $\mathcal{F}$ is acyclic (for the direct limit Čech cohomology) on each U_i, then the direct limit map $\check{H}^n(\mathcal{F}, \{U_i\}) \to \check{H}^n(X, \mathcal{F})$ is an isomorphism.

We will mostly talk about Čech cohomology here, because that's what we can write down. But there is an issue here about whether Čech cohomology coincides with sheaf cohomology (defined in terms of injective resolutions). [BGR] simply ignores this issue entirely and pretends that sheaf cohomology does not exist. (How barbaric.) [FvdP] comments that "for the G-topologies considered in this book, one can show that for all abelian sheaves the H^n coincide with the $\check{H}^n$", referring to van der Put's paper "Cohomology of affinoid spaces". I don't plan to spend any time on this point now, but I may touch on it when we talk about the cohomology of coherent sheaves later.

One thing you have to beware of when working with G-topologies is that you can't lean on the crutch of defining things in terms of stalks. This is more clear for Grothendieck topologies, when there may not be any "points" to speak of at all; but even for G-topologies, and in particular for the ones in rigid geometry, there are not enough points readily available to distinguish sheaves. That is, you can write down a nonzero abelian sheaf whose stalks are all zero. For instance, this shows up when you try to define exactness. A *short exact sequence* of abelian sheaves is a sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

such that

(a) For any admissible U, the sequence

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$$

is exact.

(b) For any admissible U and any $s \in \mathcal{F}_3(U)$, there is an admissible covering $\{U_i\}$ of U such that the restriction of s to each U_i is the image of some element of $\mathcal{F}_2(U_i)$.

Another way to say it is that $\mathcal{F}_1$ is the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$, and $\mathcal{F}_3$ is the cokernel of $\mathcal{F}_1 \to \mathcal{F}_2$. The asymmetry arises for the usual reason: given a map between sheaves, its kernel in the category of presheaves is again a sheaf, but its cokernel in the category of presheaves need not be a sheaf and so must be sheafified to get the cokernel in the category of sheaves.

More about this when we come back to considering sheaf cohomology more systematically.

A G-topology on $\mathbb{P}$

The *weak G-topology* on $\mathbb{P}$ is the G-topology in which the admissible sets are $\emptyset$, $\mathbb{P}$, and the affinoid subsets, and a covering (of an admissible sets by admissible sets) is admissible if it contains a finite subcovering.

The principal virtue of this topology is that the presheaf $\mathcal{O}$ defined by

$$\mathcal{O}(F) = A_F$$

turns out to be a sheaf.

Lemma 2. *Let $\mathcal{F}$ be a presheaf on $\mathbb{P}$ such that*

(a) If the affinoid U is a disjoint union of connected affinoids $U_1, \ldots, U_n$, then $\mathcal{F}(U) \to \oplus_i \mathcal{F}(U_i)$ is a bijection.

(b) If U_1, U_2 are connected affinoids and $U_1 \cap U_2$ is nonempty, then

$$0 \to \mathcal{F}(U_1 \cup U_2) \to \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \xrightarrow{d^0} \mathcal{F}(U_1 \cap U_2)$$

is exact.

Then $\mathcal{F}$ is a sheaf. Moreover, if d^0 is always surjective in (b), then $\check{H}^m(\mathcal{F}, \{U_i\}) = 0$ for any $m > 0$ and any admissible covering $\{U_i\}$.

Proof. This is completely formal, so I'll leave it to the reader. Or see [FvdP, Proposition 2.4.6] (although they don't do all the details either). □

Proposition 3. *The presheaf $\mathcal{O}$ on $\mathbb{P}$ is a sheaf, and its higher Čech cohomologies vanish.*

Proof. Condition (a) from Lemma 2 follows from the structure theorem we proved for A_F. (Or rather, it follows from the reduction to the connected case, for which we need the calculation at the top of the handout.) To check condition (b) and the surjectivity of d^0, we may assume that $\infty \in U_1 \cap U_2$ (by applying a fractional linear transformation); then we can replace each sheaf with the subsheaf of functions vanishing at ∞ without affecting the exactness. Then the exactness of the new sequence follows from the Mittag-Leffler decomposition (i.e., from the partial fractions decomposition of a rational function); see exercises from previous handout. □

This is a special case of a much more general theorem of Tate, which we will return to soon.

Exercises

1. Extend Proposition 1 to the case where there is no element of $|K^*|$ between r_1 and r_2. (Hint: replace $x - a$ with a separable irreducible polynomial whose roots fall between the two discs, and find a good starting g using Lagrange interpolation.)

2. Write down a sheaf on $\mathbb{P}$, equipped with the weak G-topology, whose stalks are all zero.

3. Show that the presheaf $\mathcal{O}^*$ on $\mathbb{P}$ given by $\mathcal{O}^*(F) = A_F^*$ is a sheaf (see [FvdP, Definition 2.5.9], but they don't check any details either).