18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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More on G-topologies, part 2

Corrections from last time

Correction 1: I flubbed the answer to Rebecca's question about bases of an ordinary topology and a G-topology. Before fixing that, I should maybe say some things more precisely.

If T is a G-topology on a set X, the finest G-topology T' on X which is slightly finer than T (which I'll also call the *strong refinement* of T, or the *strong topology* slightly finer than T) is given as follows [BGR, Lemma 9.1.2/3]. (This was left as an exercise in earlier notes, but I said it in class; however, I think I forgot the last restriction in (a).)

(a) A T'-open is any set which can be covered set-theoretically (i.e., is the union, *not* just a subset of the union!) by T-opens, in such a way that the restriction to any T-open can be refined to a T-covering.

(b) A T'-covering of a T'-open U is any covering by T'-opens which, when restricted to a T-open $V \subset U$, becomes a covering which is refined by some T-covering. (In particular, coverings by T-opens are T'-admissible if and only if they satisfy the condition in (a).)

In particular, any T'-open admits an admissible cover by T-opens.

Various additional things you might expect to be true about T are only true about T'. (See [BGR, 9.1.4] for verifications.)

- If $\{U_i\}$ is a T'-covering of X, then $U \subseteq X$ is T'-admissible if and only if $U \cap U_i$ is T'-admissible for each i.
- Any set-theoretic covering which can be refined to a T'-covering is also a T'-covering.
- If $\{U_i\}$ is a T'-covering of X, then a set-theoretic covering of X is a T'-covering if and only if its restriction to each U_i is a T'-covering.
- Adding some T'-opens to a T'-covering yields another T'-covering (i.e., T' is "saturated"). That's because the new cover is refined by the old cover!

If you start with an ordinary topology and make a corresponding G-topology T in which admissibles are the opens in the usual topology and coverings are open coverings, then T'-opens are already T-open because the union of opens in an ordinary topology is open. And any T'-covering is already a set-theoretic covering by opens, so already occurs in T.

So yes, you *can* recover the same G-topology by starting with a basis B of the ordinary topology which is closed under intersections, taking only basis sets to be admissible, and taking set-theoretic coverings among them.

Correction 2: Abhinav points out that my proof that a nonzero function on a connected affinoid subset has only finitely many zeroes is bogus. (I tried to reduce to the disc case by writing a connected affinoid as an intersection of discs, but the function you chose need not extend to any of those larger discs.) The proof of this is actually a real headache given what we know right now; see [FvdP, Theorem 2.2.9]. It will be a bit easier once we talk about analytic subsets; see below.

Bigger correction: the G-topology on $\mathbb{P}$

There's a more serious problem coming up: the G-topology I described on $\mathbb{P}$ is not quite the right one, because I was sloppy in defining affinoid subsets. (In fact, it was probably a bad pedagogical idea to even call them "affinoid subsets". I blame [FvdP] for leading me astray.) Here's a better definition to use in general.

A *rational subset* of $\mathbb{P}$ is one defined by the inequalities $|f_i(x)| \leq 1$ for some rational functions $f_1, \dots, f_n \in K(x)$. This allows some sets I excluded before; for instance, you can take a nonrational point and its conjugates, and take the union of the discs of some radius about those points. The right "weak G-topology" on $\mathbb{P}$ is the one in which all rational subsets are admissible, and admissible covers are those containing a finite subcover.

By the way, note that this means that rational subsets are by fiat quasicompact in the weak G-topology.

Another G-topology on $\mathbb{P}$

Remember that I mentioned (see previous exercises) that there is always a finest topology slightly finer than any given G-topology. I now want to make that topology explicit on $\mathbb{P}$.

Let T be the finest topology slightly finer than the G-topology introduced last time. Then the T-opens are precisely the opens in the usual topology on $\mathbb{P}$, since every open neighborhood of a point contains a rational subset. (Here's where you need to fix the definition that I gave last time.)

A cover $\{U_i\}$ of some U is T-admissible if for any rational $F \subset U$, there exists a finite subset J of I and rational subsets $F_j \subset U_j$ for $j \in J$ such that $F \subset \cup_{j \in J} F_j$.

We now have the notion of an "analytic function" on any open subset F of $\mathbb{P}$, namely a section of $\mathcal{O}$. More explicitly, an analytic function on F can be viewed as a limit of a sequence of rational functions which is uniform (i.e., converges under the supremum norm) on each rational subset of F. (Compare the construction of analytic functions on a complex domain as limits of polynomials which are uniform on compacts.) That means we've given F the structure of a "G-ringed space", i.e., a space with a G-topology and a sheaf of rings for the topology. In fact, F is a *locally G-ringed space*, that is, the stalks of $\mathcal{O}$ at each $x \in F$ are local rings (because the stalk of $\mathcal{O}$ at x doesn't depend on the particular choice of F).

In general, when starting with an affinoid space, its strong G-topology will be generated by "affinoid subspaces", which can be very complicated to describe. On $\mathbb{P}$, however, it will turn out that the only such spaces will be the rational ones, which explains why we get a reasonable description above.

Examples

For any $r_1 < r_2$, the cover of $\mathbb{P}$ by the discs

$$\{x \in \mathbb{P} : |x| \leq r_2\} \qquad \{x \in \mathbb{P} : |x| \geq r_1\}$$

is admissible. However, the cover by the disjoint discs

$$\{x \in \mathbb{P} : |x| < 1\}, \qquad \{x \in \mathbb{P} : |x| \geq 1\}$$

is *not* admissible. To check this, pick a closed disc $D_r : |x| \leq r$ with $r > 1$; if our original cover were admissible, then the disc $|x| < 1$ and the annulus $1 \leq |x| \leq r$ form an admissible cover of D_r. But if that were the case, I'd be able to pick out a rational subspace of $|x| < 1$ which together with the annulus covers D_r, but that's clearly imposible since the disc and the annulus form a disjoint cover of D_r and the open disc is not itself rational.

An example of an admissible cover of the open unit disc $D = \{x \in \mathbb{P} : |x| < 1\}$: let $r_1 < r_2 < \cdots$ be a sequence of elements of $r \in (0,1) \cap \Gamma$ (where again Γ is the divisible closure of $|K^*|$) increasing to 1. Then the discs $D_i = \{x \in \mathbb{P} : |x| \leq r_i\}$ form an admissible cover of D. In fact, this cover has the following interesting property: the maps $\mathcal{O}(D_i) \to \mathcal{O}(D_{i+1})$ have dense image for all i (because already $K[x]$ is dense in each $\mathcal{O}(D_i)$). A space admitting an admissible cover by a sequence of affinoids with this density property is called a *quasi-Stein space*; they turn out to have cohomological properties similar to those of affinoid spaces (and those of Stein spaces in complex analysis).

Exercises

The purpose of these exercises is to work out some details for a special class of affinoid subsets of $\mathbb{P}^1$, namely open annuli. Throughout, choose $r_1 < r_2$ and put $X = \{x \in \mathbb{P} : r_1 < |x| < r_2\}$.

1. Prove that $\mathcal{O}(X)$ consists of Laurent series series $\sum_{n \in \mathbb{Z}} c_n x^n$ over K such that

$$\lim_{n \to \pm\infty} |c_n| r^n = 0 \qquad (r_1 < r < r_2).$$

   (Hint: consider an admissible cover by closed subannuli, as in my last example.)

2. Prove that a function $\sum c_n x^n \in \mathcal{O}(X)$ is bounded if and only if $|c_n| r_1^n$ remains bounded as $n \to -\infty$ and $|c_n| r_2^n$ remains bounded as $n \to +\infty$.

3. Suppose $f \in \mathcal{O}(X)$ is bounded on X. Prove that f is equal to a polynomial times a unit of $\mathcal{O}(X)$, and deduce that f has finitely many zeroes on X.

4. Prove that $\mathcal{O}(X)$ is not a Noetherian ring, by exhibiting an ideal which is not finitely generated. In particular, the boundedness hypothesis in the previous exercise is absolutely necessary! (Hint: consider functions which vanish on all but finitely many of some infinite but non-accumulating sequence of points.)

5. Suppose that K is spherically complete. Prove that $\mathcal{O}(X)$ is a *Bézout ring*, a ring in which each finitely generated ideal is principal (even though $\mathcal{O}(X)$ is not typically Noetherian). In fact, this property is equivalent to the spherical completeness of K, but you don't have to check this. (Hint: try the case where K is discretely valued first. This result is a theorem of Lazard; see IHES 14 (1962) 47–75, or [FvdP, Theorem 2.7.6].)