18.727, Topics in Algebraic Geometry (rigid analytic geometry)
Kiran S. Kedlaya, fall 2004
Kiehl's finiteness theorems

References: [FvdP, Chapter 4]. Again, Kiehl's original papers (in German) are: Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedische Funktionentheorie, *Inv. Math.* **2** (1967), 191–214; and Theorem A und B in der nichtarchimedische Funktionentheorie, *Inv. Math.* **2** (1967), 256–273.

Addendum: Generic fibres more generally

Jay pointed out a reference for a more general construction of generic fibres: de Jong, Crystalline Dieudonné theory via formal and rigid geometry (see Chapter 7, where he attributes the construction to Berthelot). Let $P = \mathrm{Spf}(R)$, where R is a quotient of a ring of the form $R\langle x_1, \dots, x_m\rangle[\![y_1, \dots, y_n]\!]$. (Note that the order of the two sets of brackets is important here!) You can then take the generic fibre of P to be the appropriate subspace of the product of the closed m-dimensional unit ball with the open m-dimensional unit ball (and its points will be the formal subschemes of P which are integral and finite flat over $\mathfrak{o}_K$). Another way to say the same thing is that you take, for each $0 < \epsilon < 1$ in the divisible closure of $|K^*|$, the subring of $R\langle x_1, \dots, x_m\rangle[\![y_1, \dots, y_n]\!]$ consisting of series convergent for $|x_1|, \dots, |x_m| \leq 1$ and $|y_1|, \dots, |y_n| \leq \epsilon$, form the quotient, and nest these affinoid spaces to get your rigid space.

Addendum: Analytification of an algebraic variety

Here's the question Andre asked last time, with my commentary on it. Suppose X is a quasi-projective *algebraic* variety over K. Embed X into some projective variety $\overline{X}$; we can then equip the closed points of $\overline{X}$ with a rigid analytic structure by picking a model of $\overline{X}$ over K and taking the rigid analytic generic fibre of the completion along the special fibre. (This is independent of the choice of the model because the rigid analytic generic fibre doesn't see blowups in the special fibre.) The closed points of X form a subspace of $\overline{X}$; the question is, is this subspace independent of the choice of $\overline{X}$?

It suffices to check that if $\overline{X}_1 \to \overline{X_2}$ is a proper morphism between two compactifications, then it induces an isomorphism on X as a rigid analytic space. (To compare two general compactifications, you can then compare each to the fibre product.) I think this is okay, but I didn't check it.

Separated and proper spaces

The category of rigid spaces has fibre products: these are generated by completed tensor products of affinoid algebras. We may thus say that a rigid space X is *separated* if the diagonal $\Delta : X \to X \times_K X$ is a closed immersion (i.e., is defined by a coherent sheaf of ideals). As for schemes, one has the following criterion for separatedness.

Proposition 1. *A rigid space X is separated if and only if it admits an admissible affinoid covering $\{U_i\}$ such that for $i \neq j$ with $U_i \cap U_j \neq \emptyset$, the intersection $U_i \cap U_j$ is affinoid and the canonical map $\mathcal{O}(U_i)\widehat{\otimes}_K \mathcal{O}(U_j) \to \mathcal{O}(U_i \cap U_j)$ is surjective.*

In particular, any affinoid space is separated.

If U is an affinoid subset of an affinoid space X, then U is said to be an *interior subspace* of X if it is contained in a subspace of the form $|f_1(x)|, \dots, |f_n(x)| \leq \epsilon$ for some $f_1, \dots, f_n \in \mathcal{O}(X)$ such that $K\langle f_1, \dots, f_n \rangle$ surjects onto $\mathcal{O}(X)$ and some $\epsilon < 1$. (Note that "the interior of X" is not a useful notion here.)

A rigid space X is *proper* if it is separated and there exist two finite admissible affinoid coverings $\{U_i\}_{i=1,\dots,n}$ and $\{U_i'\}_{i=1,\dots,n}$ such that U_i is an interior subspace of U_i' for $i = 1, \dots, n$. For instance, $\mathbb{P}^n_K$ (with homogeneous coordinates $t_0, \dots, t_n$) is proper because you can cover it by the subspaces

$$U_{i,\epsilon} = \{x \in \mathbb{P}^n_K : |t_j(x)| \leq \epsilon |t_i(x)|\} \qquad (i = 0, \dots, n)$$

for any ϵ, and $U_{i,\epsilon}$ is an interior subspace of $U_{i,\epsilon'}$ if $\epsilon < \epsilon'$. Likewise, any closed analytic subspace of a proper space is proper, so the analytic space associated to a projective variety is also projective.

The relative version of this construction is as follows. A morphism $X \to Y$ is proper if after restricting Y to each element of some admissible affinoid cover, I can find two finite admissible affinoid coverings $\{U_i\}$ and $\{U_i'\}$ of X, such that U_i is a relative interior subspace of U_i'. The latter means that U_i belongs to a subspace of U_i' of the form $|f_1(x)|, \dots, |f_n(x)| \leq \epsilon$ for some $f_1, \dots, f_n \in \mathcal{O}(X)$ such that $\mathcal{O}(Y)\langle f_1, \dots, f_n \rangle$ surjects onto $\mathcal{O}(U_i')$ and some $\epsilon < 1$. (This is admittedly a lousy definition; Kiehl rigged it up precisely to make the next theorem work. Is there a "universally closed" version of this definition, or a valuative criterion?)

A morphism $f : X \to Y$ is *finite* if for some admissible affinoid covering $\{U_i\}$ of Y, each $f^{-1}(U_i)$ is affinoid and $\mathcal{O}(U_i) \to \mathcal{O}(f^{-1}(U_i))$ is a finite morphism of affinoid algebras. One can recover f from the pushforward sheaf $f_* \mathcal{O}_X$, which is coherent; in particular, one can show that "some" may be replaced by "any" (see [FvdP, Definition 4.5.7]). Any finite morphism is proper, because $f^{-1}(U_i)$ is a relative interior subspace of itself!

Theorem 2 (Kiehl). *(a) Let X be a proper rigid space over K. Then the (Čech) cohomology spaces of any coherent sheaf on X are finite dimensional over K.*

(b) Let $f : X \to Y$ be a proper morphism of rigid spaces, with Y separated, and let $\mathcal{F}$ be a coherent sheaf on X. Then the direct image $f_* \mathcal{F}$ and the higher direct images $R^i f_* \mathcal{F}$ are coherent sheaves on Y. In particular, the image $f(X)$ is a closed analytic subspace of Y.

Warning: the "higher direct images" here are constructed using Čech cohomology; I think you should be able to make a Čech/sheaf comparison in this relative case, but I didn't check it.

Sketch of proof. This sketch is basically the sketch from [FvdP, Theorem 4.10.3]. The idea is to take the two coverings you have and show that, on one hand, you get the same Čech cohomology groups (by acyclicity) from both coverings, and on the other hand, the map $\mathcal{F}(U) \to \mathcal{F}(U')$ is a compact operator (a uniform limit of operators of finite rank) whenever U is an interior subspace of U'. Thus on each cohomology space, the identity map is compact, which can only happen on a finite dimensional vector space. (The relative argument proceeds basically the same way.) □

I don't know a reference for "rigid GAGA", but I seem to think it was written down somewhere by Kiehl. Anyway, [FvdP] says you can just imitate Serre's proof; rather than presume you know how this goes, I'll give a brief version here. (Serre's paper is as good a reference as any, maybe even better, because it's from the dark ages when Serre's "Faisceaux algébriques coherents", which introduced sheaves into abstract algebraic geometry, was hot off the presses. It is: Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier (Grenoble)* **6** (1955–56), 1–42. His Théorèmes 1,2,3 are our (a),(b),(c).)

Theorem 3 (Rigid GAGA). *Let X be a projective algebraic variety over K.*

- *(a) For any algebraic coherent sheaf $\mathcal{F}$ on X, the natural homomorphisms from algebraic (sheaf) to analytic (Čech) cohomologies are bijections.*
- *(b) The analytification functor from coherent sheaves on X to coherent sheaves on the analytification of X is fully faithful.*
- *(c) Every analytic coherent sheaf comes from an algebraic coherent sheaf (so the functor in (b) is an equivalence of categories).*

Sketch of proof. Note that it's enough to prove everything for $X = \mathbb{P}^n$, since we can extend coherent sheaves by zero in both the algebraic and analytic categories to get from sheaves on X to sheaves on $\mathbb{P}^n$.

For (a), you first prove it for $\mathcal{O}$, then for the sheaves $\mathcal{O}(m)$ by induction on the dimension n of the projective space: if H is a hyperplane, you have an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$$

where the first map is multiplication by a linear form vanishing on H. Then clever use of the five lemma gives you what you need. Since every coherent sheaf on $\mathbb{P}^n$ is the kernel of a map from some $\mathcal{O}(m)$ to some $\mathcal{O}(n)$ (another theorem of Serre!), you end up getting (a) for all coherent sheaves.

For (b), you apply (a) to the sheaf Hom between any two given algebraic sheaves: homomorphisms between the sheaves correspond to elements of H^0 of the sheaf hom.

For (c) (the hard part), you again induct on dimension. The crux of the argument is to show that given $\mathcal{F}$ and a point $x \in \mathbb{P}^n$, you can find an integer m such that the analytic H^0 of the twist $\mathcal{F}(m)$ generates the stalk of $\mathcal{F}(m)$; by a compactness argument, you can choose m uniformly for all x. That is, all stalks of $\mathcal{F}(m)$ are generated by the global sections;

proceeding as in the proof of Kiehl's theorem on coherent sheaves on affinoid spaces, we deduce that $\mathcal{F}$ can be written as a cokernel of a map between algebraic coherent sheaves (both of the form $\mathcal{O}(-m)^d$), and by (b) is algebraic.

The crux lemma is a bit intricate; it's Lemme 8 in Serre's paper. You again pick a hyperplane H and write down the exact sequence

$$0 \to \mathcal{O}(H) \cong \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0,$$

which on the right you can tensor with $\mathcal{F}$:

$$\mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0;$$

let $\mathcal{K}$ be the kernel on the left. Both $\mathcal{K}$ and $\mathcal{F}_H$ are supported on H, so are algebraic by the induction hypothesis; in particular, by Serre's theorem, they lose their higher cohomology upon twisting by $\mathcal{O}(m)$ for m sufficiently large. If you split the four-term exact sequence

$$0 \to \mathcal{K}(m) \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{F}_H(m) \to 0$$

by adding $\mathcal{G}$ in the middle, you end up with surjections

$$H^1(\mathbb{P}^{n,\mathrm{an}}, \mathcal{F}(m-1)) \to H^1(\mathbb{P}^{n,\mathrm{an}}, \mathcal{G}), \qquad H^1(\mathbb{P}^{n,\mathrm{an}}, \mathcal{G}) \to H^1(\mathbb{P}^{n,\mathrm{an}}, \mathcal{F}(m));$$

the point is that $\dim H^1(\mathbb{P}^{n,\mathrm{an}}, \mathcal{F}(m))$ is nonincreasing as m grows. It must thus stabilize after some m, at which point one finds that the global sections of $\mathcal{F}(m)$ surject onto those of $\mathcal{O}_H(m)$. The latter generate the stalk of $\mathcal{F}_H(m)$ at x for m large enough (because this is true for the algebraic stalk, and the map between the algebraic and analytic local rings is faithfully flat—they have the same completion), so you win. $\square$

Quasi-Stein spaces

I would be remiss in not mentioning Kiehl's other big cohomological theorem, his analogue of Cartan's "Theorem A" and "Theorem B" in complex analysis.

A rigid space X is *quasi-Stein* if it admits an admissible affinoid covering $U_1 \subseteq U_2 \subseteq \cdots$ in which the image of $\mathcal{O}(U_{i+1})$ is dense in $\mathcal{O}(U_i)$ for each i. Besides affinoid spaces themselves, examples include open balls, one-dimensional annuli, and products of other quasi-Stein spaces.

Here's Kiehl's main theorem about quasi-Stein spaces. Kiehl's argument is a bit fragmentary (see below), so I'm looking for a better reference (preferably not in German); suggestions?

Theorem 4 (Kiehl). *Let $\mathcal{F}$ be a coherent sheaf on a quasi-Stein space X, with a covering $\{U_i\}$ as above.*

(a) The image of $\mathcal{F}(X)$ in $\mathcal{F}(U_i)$ is dense for all i.

(b) The cohomology groups $H^i(X, \mathcal{F})$ vanish for all i ("Theorem B"). Note that van der Put's theorem applies, so sheaf = Čech here.

(c) For each $x \in X$, $\mathcal{F}_x$ is generated as an $\mathcal{O}_x$-module by global sections of $\mathcal{F}$ ("Theorem A").

Proof. Note that $\mathcal{F}(U_{i+1})$ is dense in $\mathcal{F}(U_i)$, from which apparently (a) is "immediate" ("unmittelbar"), but I don't see why offhand. (Do you?) To prove (b), it's enough to prove H^1 always vanishes (by a dimension shifting argument); this is done by an explicit calculation (see p. 271 of Kiehl's "Theorem A und B" paper). To prove (c), let $\mathcal{G}$ be the ideal sheaf of x; since $H^1(X, \mathcal{F} \otimes \mathcal{G})$ vanishes, the sequence

$$\mathcal{G}(X) = H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}/(\mathcal{F} \otimes \mathcal{G})) = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \to 0$$

is exact, and (c) follows by Nakayama's lemma. □

This *almost* means you can pretend quasi-Stein spaces are just like affinoid spaces from the point of view of the cohomology of coherent sheaves. However, note that (c) does not imply that $\mathcal{F}$ is generated by *finitely many* global sections, because there is no compactness argument available. And indeed, it may not be: e.g., let $P_1, P_2, \dots$ be points in the open unit disc such that $|P_i| \to 1$ as $i \to \infty$, and take the direct sum of the ideal sheaves of the P_i.

On the other hand, it sometimes happens that all *locally free* coherent sheaves are generated by finitely many global sections even though the underlying space is not affinoid, e.g., on an open annulus (see exercise from an earlier handout).