18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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The Lubin-Tate moduli space

Note: there is no rigid geometry in this set of notes! That will come next time, when we talk about the period mapping.

References: For starters, the original paper of Lubin-Tate (which involves no rigid geometry, only formal geometry) is: J. Lubin and J. Tate, Formal moduli for one-parameter formal Lie groups, *Bulletin de la Soc. Math. de France*, **94** (1966), 49–59. (This is not their paper on local class field theory, though of course the two are closely related.) As for "Gross-Hopkins", there are two such papers. One (here [GH1]) is: M.J. Hopkins and B.H. Gross, The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory, *Bulletin of the AMS* **30** (1994), 76–86. This paper makes the link between Lubin-Tate spaces and stable homotopy theory; I won't do that here. It is quite cursory on the geometry side (I'm not capable of judging on the homotopy side), to the point of being barely legible. The second paper (here [GH2]) is: M.J. Hopkins and B.H. Gross, Equivariant vector bundles on the Lubin-Tate moduli space, in *Topology and Representation Theory*, Contemporary Mathematics **158**, AMS, 1994, 23–88. As the page count suggests, this is *much* more detailed and focuses entirely on the rigid geometry.

Formal groups

Before we do any geometry, here's a quick review of formal groups. The standard reference is Hazewinkel, *Formal Groups*, though I think his formulas have some errors in them; use with caution.

A *(commutative) formal group of dimension* n over a ring R is a (commutative) cogroup structure on $R[[x_1, \dots, x_n]]$ with identity 0, i.e., a (commutative) comultiplication satisfying the usual (co)group axioms. It's of course enough to specify how $x_1, \dots, x_n$ behave under the comultiplication; their images form an n-tuple of power series

$$F(X, Y) = (F_1(X, Y), \dots, F_n(X, Y))$$

(where X is short for $x_1, \dots, x_n$ and Y for $y_1, \dots, y_n$), such that

$$\begin{gathered} F(X, 0) = X, \quad F(0, Y) = Y \\ F(X, F(Y, Z)) = F(F(X, Y), Z) \\ F(X, Y) = F(Y, X). \end{gathered}$$

(Exercise: the existence of inverses is automatic given the other axioms.) A morphism $f : F \to F'$ of formal groups is an n-tuple of n-variate power series such that $F'(f(X), f(Y)) = F(X, Y)$. Let $\mathrm{Lie}(F)$ denote the (relative) tangent space of $R[[x_1, \dots, x_n]]$, i.e., the trivial Lie algebra over R generated by $x_1, \dots, x_n$. Every endomorphism of F induces a R-linear endomorphism on $\mathrm{Lie}(F)$, which is just look at the linear terms in the power series.

Examples: take any algebraic group of dimension n and restrict the group law to the tangent space at the origin, and you get a formal group of dimension n. For $\mathbb{G}_a$, you get

$$F(x,y) = x + y.$$

For $\mathbb{G}_m$, you get

$$F(x,y) = x + y + xy,$$

or $x + y - xy$ for another choice of coordinates, or crazier things for more bizarre choices of coordinates. (Those two are isomorphic if $\mathbb{Q} \subseteq R$, even though the original groups are not, but not in general. I'll make a stronger statement below.) Any elliptic curve gives a formal group of dimension 1, as constructed in Silverman's book. Also, abelian varieties and linear groups give you other examples in higher dimension; however, I'm mostly interested here in dimension 1.

If I knew what it was, I would mention here the connection between formal groups of dimension 1 and stable homotopy theory (which as far as I can tell is due more or less entirely to Hopkins). However, I don't; maybe Mark can enlighten us a bit at some point.

Formal $\mathfrak{o}$-modules

I'm also going to work a bit (following [GH2]) with Drinfeld's more general notion of formal $\mathfrak{o}$-modules, where $\mathfrak{o}$ is a complete DVR with finite residue field $k = \mathbb{F}_q$. Fix a choice of a uniformizer π of $\mathfrak{o}$, and put $K = \operatorname{Frac} \mathfrak{o}$ as usual. Let R be a (commutative) $\mathfrak{o}$-algebra; I'll call the structure map $i : \mathfrak{o} \to R$ if I need to refer to it. A *formal* $\mathfrak{o}$*-module of dimension* n *over* R is a (commutative) formal group F of dimension n equipped with a ring homomorphism $\theta = \theta_F : \mathfrak{o} \to \operatorname{End}_R(F)$, such that

$$\theta(a)(X) \equiv i(a)X \pmod{(x_1, \dots, x_n)^2}.$$

(That is, the action of $\theta(a)$ on $\operatorname{Lie}(F)$ is by multiplication by $i(a)$.) A formal group is automatically a formal $\mathbb{Z}_p$-module as long as $\mathbb{Z}_p \subseteq R$.

Convention: I'll write a_F instead of $\theta_F(a)$, as in [GH2].

Example: the first Lubin-Tate paper (Formal complex multiplication in local fields, *Ann. Math.* **81** (1965), 380–387) show that you can uniquely specify a formal $\mathfrak{o}$-module of dimension 1 by specifying the action of π: it must be given by a series $f(x)$ with $f(x) \equiv \pi x \pmod{x^2}$ and $f(x) \equiv x^q \pmod{\pi}$.

Example: in equal characteristic p, Drinfeld stumbled across these as the analogues of the formal group associated to an algebraic group, for what we call "Drinfeld modules". Briefly, a Drinfeld module is an action of a finite extension of the ring $\mathbb{F}_q[t]$ on the additive group of a ring in characteristic p, via "additive polynomials":

$$x \mapsto c_0 x + c_1 x^p + \cdots + c_n x^{p^n}.$$

These act like abelian varieties in many ways (e.g., producing Galois representations), but have much simpler moduli and so are useful for things like proving the Langlands correspondence for GL_2 over function fields (Drinfeld's original application).

You can speak of the "invariant differentials" of F, i.e., the elements ω of the module of formal differentials (i.e., the free $R[\![X]\!]$ over $x_1, \dots, x_n$) such that $\omega(F(X,Y)) = \omega(X)+\omega(Y)$, and $\omega(a_F(X)) = i(a)\omega(X)$ for $a \in \mathfrak{o}$. These form a free R-module of rank n, called $\omega(F)$; in fact, the "quotient mod degree 2" map from invariant differentials to $R\,dx_1 \oplus \cdots \oplus R\,dx_n$ is a bijection, and all invariant differentials are closed [GH2, Proposition 2.2].

Policy: I'm now going to assume dimension 1 forever after, because Lubin-Tate theory applies only in dimension 1. Also, I may skip the $\mathfrak{o}$-module generalizations of some statements about formal groups, but those are all straightforward to extend (or see [GH2]).

Logarithms

If $f : F \to \mathbb{G}_a$ is a homomorphism of formal $\mathfrak{o}$-modules, you can take its formal derivative

$$\omega = df(x) = \frac{df}{dx} \cdot dx.$$

That gives a homomorphism $d : \mathrm{Hom}(F, \mathbb{G}_a) \to \omega(F)$. By [GH2, Proposition 3.2], if R is flat (i.e., torsion-free) over $\mathfrak{o}$, then d is injective; if R is a K-algebra, then d is bijective. In particular, in the latter case, there is a unique isomorphism $f : F \to \mathbb{G}_a$ with df equal to any prescribed generator of $\omega(F)$. We call f a *logarithm* for F.

Height

It's an easy lemma [GH2, Lemma 4.1; beware that f is used to mean two different things in the same sentence!] that if R is a field and F is a formal group with $i(\pi) = 0$, then either $\pi_F = 0$ or there is an integer h such that

$$\pi_F(x) = f(x^{q^h})$$

for some series f with $f'(0) \neq 0$. In the second case, we say F has *height h*. If R is a complete local ring whose maximal ideal I contains $i(\pi)$ (hereafter a *local* $\mathfrak{o}$*-algebra*), we say F has height h if the reduction of F has height h over R/I. (Define height for a formal $\mathfrak{o}$-module as the height of the underlying formal group. Oh, and this definition doesn't depend on dimension 1.)

Deformations

If we fix a formal $\mathfrak{o}$-module F_0 of some height h over R/I, I will refer to a formal $\mathfrak{o}$-module F over R equipped with an isomorphism of its reduction to F_0 as a *deformation of* F_0 *over* R. Then the Lubin-Tate-Drinfeld theorem explicitly describes a universal deformation of F_0. This comes from the following fact: if R is flat over $\mathfrak{o}$, then any formal $\mathfrak{o}$-module (of dimension 1, as always here) can be presented so that its formal logarithm takes the form

$$f(x) = x + \sum_{k=1}^{\infty} b_k x^{q^k}$$

for $b_k \in R \otimes K$, and this presentation is unique. A formal $\mathfrak{o}$-module presented this way is said to be $\mathfrak{o}$-*typical.* (See [GH2, §5].)

Example: for $\mathbb{G}_m$ over $\mathbb{Z}_p$, $b_k = p^{-k}$ and you get the formal logarithm of the Artin-Hasse exponential.

As the previous example shows, it's better to work with a certain change of variable here. Keeping R flat, define $v_1, v_2, \dots$ by

$$\pi b_k = v_k + b_1 v_{k-1}^q + \cdots + b_{k-1} v_1^{q^{k-1}};$$

then the v_k turn out to be integral. In fact, unwinding the construction yields a "universal formal $\mathfrak{o}$-typical module" over the infinite polynomial ring $\mathfrak{o}[v_1, v_2, \dots]$.

It turns out that you can read off heights easily here: the $\mathfrak{o}$-typical module constructed above has height h if and only if $v_1, \dots, v_{h-1}$ vanish in R/I and v_h does not.

The Lubin-Tate-Drinfeld theorem (Lubin-Tate for formal groups, Drinfeld for formal $\mathfrak{o}$-modules) now asserts that if you pull back the universal formal $\mathfrak{o}$-typical module to $\mathfrak{o}[\![u_1, \dots, u_{h-1}]\!]$ via

$$\begin{aligned} v_i &\mapsto u_i \qquad (i = 1, \dots, h-1) \\ v_h &\mapsto 1 \\ v_i &\mapsto 0 \qquad (i \geq h+1), \end{aligned}$$

and call the result F, then F is a universal deformation of its reduction modulo $(\pi, u_1, \dots, u_{h-1})$ (which is thus defined over $\mathbb{F}_q$, and which has height h). See [GH2, Proposition 12.10] for the (easy) cohomological computation that verifies this.