18.727, Topics in Algebraic Geometry (rigid analytic geometry)
Kiran S. Kedlaya, fall 2004
Periods on the Lubin-Tate moduli space

To keep things moving, I'm going to be terse now (as in [GH1]); there are lots of details filled in in [GH2].

References: Jay notes that [GH1] (the terse one) is available online; I'll put a link on the notes page.

Corrections from last time

Thanks to Jay for these.

page 2: the first Lubin-Tate paper only deals with height 1 (that is all that is needed for local class field theory), and they show uniqueness only over the completion of the maximal unramified extension of K. (Indeed, the fact that they are not isomorphic is key to being able to use them for explicit local reciprocity!)

page 3, top of page: "free $R[\![X]\!]$ over $x_1, \dots, x_n$" should be over $dx_1, \dots, dx_n$. By $\omega(F(X,Y))$, I mean you write $\omega(X) = f_1(X)dx_1 + \cdots + f_n dx_n$, then you write $F(X,Y) = (F_1(X,Y), \dots, F_n(X,Y))$, you plug in

$$\omega(F(X,Y)) = f_1(F(X,Y))dF_1(X,Y) + ... + f_n(F(X,Y))dF_n(X,Y),$$

then expand each $dF_i(X,Y)$ by the chain rule.

page 3, Height: change "formal group" in the second line to "formal $\mathfrak{o}$-module" and scratch the reference to formal $\mathfrak{o}$-modules in the parenthetical.

A group action

Last time, we built a universal deformation F over $A = \mathfrak{o}[\![v_1, \dots, v_{h-1}]\!]$ of a formal $\mathfrak{o}$-module over $\mathbb{F}_q$ of height h, which I'll denote by F_0. That means that the group $G = \operatorname{Aut}(F_0)$ acts on the deformation space $\operatorname{Spf} A$, and on the corresponding rigid analytic space X, which is the open unit polydisc in $v_1, \dots, v_{h-1}$.

It turns out that $D = \operatorname{End}(F_0)$ is a division algebra of degree n, G is the group of units in some maximal order therein, and $D \otimes K \cong M_n(K)$ is split. That means G has a natural n-dimensional linear representation V_K over K, as does D^*. In particular, G acts on the hyperplanes of V_K, i.e., on $\mathbb{P}(V_K^\vee)$; the latter carries a rigid analytic space structure, and the group action is by analytic morphisms.

The crystalline period mapping, to be defined, is a rigid analytic G-equivariant étale morphism $\Phi : X \to \mathbb{P}(V_K^\vee)$ which classifies deformations "up to isogeny" as follows. For A an affinoid algebra over K, let F_a and F_b be deformations of F_0 over $\mathfrak{o}_A$, corresponding to points $a, b \in X(A)$. Then an isogeny of F_0, viewed as an element $T \in D^*$, deforms to an isogeny $F_a \to F_b$ if and only if $T\Phi(a) = \Phi(b)$.

The universal additive extension

In order to specify Φ, I have to give you a G-equivariant line bundle $\mathcal{L}$ on X and a $K[G]$-homomorphism $V_K \to H^0(X, \mathcal{L})$ whose image is basepoint-free (i.e., the images don't all vanish at a point). For $x \in X$, we then take $\Phi(x)$ to be the hyperplane of V_K which maps to sections of $\mathcal{L}$ vanishing at x.

First, $\mathcal{L}$ is the inverse of the analytification of the sheaf ω of invariant differentials, a/k/a the Lie algebra $\mathrm{Lie}(F)$. In order to make the map, I must consider the *universal additive extension* E of F; it sits in an exact sequence

$$0 \to N \to E \to F \to 0$$

with $N = \mathbb{G}_a \otimes \mathrm{Ext}(F, \mathbb{G}_a)^\vee$, and it is universal: if $0 \to N' \to E' \to F \to 0$ is an extension of F by an additive $\mathfrak{o}$-module, then there are unique homomorphisms $i : E \to E'$, $j : N \to N'$ such that

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "j"] & E \arrow[r] \arrow[d, "i"] & F \arrow[r] \arrow[d, "\mathrm{id}_F"] & 0 \\
0 \arrow[r] & N' \arrow[r] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

commutes. (This is straightforward, modulo some cohomology arguments which have already been exploited in constructing the universal deformation, namely, that $\mathrm{Hom}(F, \mathbb{G}_a) = 0$ and $\mathrm{Ext}(F, \mathbb{G}_a)$ is free of rank $n-1$. See [GH2, Proposition 11.3].) On the level of Lie algebras, we have

$$0 \to \mathrm{Lie}(N) \to \mathrm{Lie}(E) \to \mathrm{Lie}(F) \to 0$$

and this sequence is G-equivariant.

The bundle $\mathrm{Lie}(E)$ turns out to be the covariant Dieudonné module of F_0, so it is an "F-crystal": it carries an integrable connection $\nabla : \mathrm{Lie}(E) \to \mathrm{Lie}(E) \otimes \Omega^1_{A/\mathfrak{o}}$ plus a "Frobenius structure". The latter can be viewed as an isomorphism $\sigma^* \mathrm{Lie}(E) \to \mathrm{Lie}(E)$ for any $\sigma : A \to A$ lifting the q-power map on the special fibre. Let $\mathcal{M}$ be the analytification of $\mathrm{Lie}(E)$, as a rigid vector bundle over X; then by "Dwork's trick", $\mathcal{M}$ admits a basis of horizontal sections over X. (The idea: by formal integration, you get a basis over a small polydisc. But then you use the Frobenius pullback to "grow" this polydisc.) If you prefer, this can all be described in formulas in this case: see [GH2, Section 22].

We now have our representation $V_K = H^0(X, \mathcal{M})^\nabla$ on the horizontal sections of $\mathcal{M}$: the surjection $\mathcal{M} \to \mathcal{L} \to 0$ gives the map $V_K \to H^0(X, \mathcal{L})$ whose image is basepoint-free. The verification that Φ is étale and detects isogenies can be found in [GH2, Section 23].

What is Dwork's trick?

This is worth explaining a bit more, because it also comes up all over the place in p-adic cohomology. Say you have a vector bundle $\mathcal{M}$ over the open unit polydisc X over K with coordinates $x_1, \dots, x_n$. (Note: what was $n-1$ before is n now for notational simplicity.) It turns out that $\mathcal{M}$ is in fact generated freely by global sections. [Correction of 27 Oct 2012:

this was previously attributed to Gruson, which is incorrect. Instead see: W. Bartenwerfer, k-holomorphe Vektorraumbündel auf offenen Polyzylindern, *J. reine angew Math.* **326** (1981), 214-240.]

Say you also have an integrable connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1$, i.e., commuting actions of $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$. You can then try to formally solve for the horizontal sections around $x_1 = \cdots = x_n = 0$. The resulting sections will converge on some polydisc, but its radius may be much smaller than 1. E.g., if $n = 1$, the rank is 1, and $\partial_i \mathbf{v} = c\mathbf{v}$, the horizontal section is $\exp(-\int c)\mathbf{v}$, which converges on some disc but possibly a small one.

What having a Frobenius structure does is give you an isomorphism between $\mathcal{M}$ and its pullback along some map σ lifting the q-power Frobenius on the reduction of $\Gamma(\mathcal{O}, X)$, e.g., one taking K into itself and taking x_i to x_i^q. That pulls back your sections on a tiny polydisc to sections on a larger polydisc (in my example, sections on the disc $|x_i| \leq \rho$ pull back to $|x_i| \leq \rho^{1/q}$); but in fact the K-vector space of horizontal sections is unique, so these sections actually extend the ones you started with. Repeat *ad infinitum*.