18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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A little p-adic functional analysis (part 1 of 2)

I'm going to start with a little bit of terminology and notation about nonarchimedean Banach spaces (trusting that you can fill in a few details that are similar to the real/complex case). There's a lot more where this came from, but we won't need the rest of it just yet.

Thanks to Abhinav Kumar for providing the corrections incorporated into this version (of 16 Sep 04).

References: [FvdP], Chapter 1; the stuff is also in [BGR], but you'll have to tease it out of Chapter 2 with some effort, as it's scattered among many sections. In case you come down with a craving for more p-adic functional analysis, I recommend *Nonarchimedean functional analysis*, by Schneider. This book is available online; see the notes page for a link. At worst, you can always pick up a standard functional analysis book (e.g., *Espaces vectoriels topologiques* by Bourbaki) and redo all the constructive proofs (i.e., skip anything involving Hahn-Banach) yourself in the nonarchimedean context!

Ultrametric spaces

An *ultrametric* (or *nonarchimedean metric*) on a set X is function $d : X \times X \to \mathbb{R}_{\geq 0}$ with the following properties.

(a) For $x_1, x_2 \in X$, $x_1 = x_2$ if and only if $|x| = 0$.

(b) For $x_1, x_2 \in X$, $d(x_1, x_2) = d(x_2, x_1)$.

(c) For $x_1, x_2, x_3 \in X$, $d(x_1, x_3) \leq \max\{d(x_1, x_2), d(x_2, x_3)\}$ (strong triangle inequality)

Note that if $d(x_1, x_2) \neq d(x_2, x_3)$, then in fact $d(x_1, x_3) = \max\{d(x_1, x_2), d(x_2, x_3)\}$, otherwise you get a contradiction by applying (c) to x_1, x_2, x_3 in another order.

A *Cauchy sequence* in X is a sequence $\{x_n\}_{n=1}^\infty$ such that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_m, x_n) < \epsilon$. Note that by the strong triangle inequality, this is equivalent to $d(x_n, x_{n+1}) < \epsilon$ for $n \geq N$; this is the first of many instances in which nonarchimedean analysis turns out to be easier than traditional analysis!

We say X is *complete* if every Cauchy sequence converges to a limit (necessarily unique because of (a)). We say X is *spherically complete* if every decreasing sequence of closed balls has nonempty intersection: that is, given $x_1, x_2, \dots \in X$ and $r_1, r_2, \dots \in \mathbb{R}_{\geq 0}$ such that the sets

$$D_n = \{x \in X : d(x, x_n) \leq r_n\}$$

satisfy $D_1 \supseteq D_2 \supseteq \cdots$, then $\cap_{n=1}^\infty D_n \neq \emptyset$. Note that if X is spherically complete, then X is also complete: given a Cauchy sequence $\{x_1, x_2, \dots\}$, we can pass to a subsequence if needed to ensure that $d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ for all n. Then the balls

$$D_n = \{x \in X : d(x, x_n) \leq d(x_{n+1}, x_n)\}$$

have nonempty intersection, which must be a limit of the sequence. (In other words, complete means spherically complete when the balls have radii going to 0.)

Ultrametric fields

An *ultrametric field*, or *nonarchimedean valued field* (in the terminology of [FvdP]), is a field K equipped with a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ with the following properties.

(a) For $x \in K$, $x = 0$ if and only if $|x| = 0$.

(b) For $x_1, x_2 \in K$, $|x_1 x_2| = |x_1||x_2|$.

(c) For $x_1, x_2 \in K$, $|x_1 + x_2| \leq \max\{|x_1|, |x_2|\}$.

The function $d(x_1, x_2) = |x_1 - x_2|$ is then an ultrametric on K, so we know what it means for K to be complete. In this course, we will usually be working over a complete ultrametric field. Oh, and there is always a trivial absolute value given by

$$|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0; \end{cases}$$

I'm always going to assume (unless otherwise specified) that my absolute value function is not the trivial one.

If K is an ultrametric field, then the set of $x \in K$ with $|x| \leq 1$ is a subring of K. I'll denote it by $\mathfrak{o}_K$, or sometimes by $\mathfrak{o}$ in case K is to be understood. I might also call it the "valuation subring". The set of $x \in K$ with $|x| < 1$ is a maximal ideal of $\mathfrak{o}_K$, which I'll denote $\mathfrak{m}_K$. The field $\mathfrak{o}_K/\mathfrak{m}_K$ is called the *residue field* of K, and I'll typically call it k. (Note that k is really a field, and not a one-element ring, because the absolute value function is nontrivial!)

We call $|K^*|$ the *value group* of K. We say K is *discretely valued* if its value group is a discrete subgroup of $\mathbb{R}_{>0}$; that means it must be isomorphic to $\mathbb{Z}$. Our favorite examples of complete ultrametric fields are discretely valued, namely:

(a) the field of formal power series (or really Laurent series, but I'll call it the "field of formal power series" from here on) $k((t))$ over a field k;

(b) the field $\mathbb{Q}_p$ of p-adic numbers;

(c) any finite extension of either of these (see exercises).

Another example is the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{unr}}$ of $\mathbb{Q}_p$ (or of a finite extension of $\mathbb{Q}_p$).

For non-discretely valued examples, keep reading. Then again, if you want to pretend for the rest of the course that all ultrametric fields are discretely valued, you will not lose too much of the flavor of the course. (I'll try to make explicit warnings at points where it makes a difference.)

Spherically complete fields

Terminology warning: the term *maximally complete* is used interchangeably with *spherically complete* when talking about ultrametric fields. (For instance, [FvdP] uses "maximally complete" consistently.) The reason: the ultrametric field K is spherically complete if and only if it is maximal among ultrametric fields with the same value group and residue field. (I think this is due to Kaplansky; see his papers "Maximal fields with valuations I, II". See also the exercises.)

Note that any discretely valued ultrametric field is spherically complete, since the radii of balls in a decreasing sequence must stabilize. The canonical example of an ultrametric field which is complete but not spherically complete is $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$.

Every ultrametric field can be embedded in a spherically complete ultrametric field, or even an algebraically closed spherically complete ultrametric field. I don't know a reference for this offhand, but see the exercises for some examples.

Nonarchimedean Banach spaces

Assume for the rest of this installment and the next (and pretty much for the rest of the course!) that K is a *complete* ultrametric field, and let $|\cdot|$ denote the norm on K.

Let V be a vector space over K. A *seminorm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following conditions.

(a) For $a \in K$ and $v \in V$, $\|av\| = |a|\|v\|$.

(b) For $v, w \in V$, $\|v + w\| \leq \max\{\|v\|, \|w\|\}$.

If moreover $\|v\| = 0$ implies $v = 0$, we say $\|\cdot\|$ is a *norm.* If V comes equipped with a norm, we call it a *normed space (over K).*

If V is a normed space which is complete under its norm (or rather, under the induced ultrametric $d(v, w) = \|v - w\|$), we say V is a *Banach space (over K).* For instance, any finite dimensional K-vector space is a Banach space. As in the traditional setting, there are lots of simple examples of Banach spaces, e.g., the set of all null sequences $(a_0, a_1, \dots)$ over K (that is, sequences with $|a_i| \to 0$ as $i \to \infty$) with the supremum norm, or all bounded sequences with the supremum norm, or all convergent sequences with the supremum norm, or...

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the same space V are *equivalent* if there exist $\alpha, \beta > 0$ such that for all $v \in V$,

$$\alpha\|v\|_1 \leq \|v\|_2 \leq \beta\|v\|_1;$$

clearly this is actually an equivalence relation on norms. Equivalent norms induce the same topology on V, but *not conversely.*

One makes subspaces and quotient spaces as follows. If $f : V \to W$ is an injective map of vector spaces over K, W is a Banach space, and $\mathrm{im}(f)$ is closed, then the restriction of the norm on W to V gives V the structure of a Banach space. If $f : V \to W$ is a surjective map of vector spaces over K, V is a Banach space, and $\ker(f)$ is closed, then the *quotient norm*

$$\|w\|_W = \inf\{\|v\| : v \in V, f(v) = w\}$$

is obviously a seminorm, but in fact it is also a norm. Namely, if $\|w\|_W = 0$, we can choose $v_1, v_2, \dots \in V$ with $f(v_i) = w$ for all i and $\|v_i\| \to 0$. Then $v_i - v_j \in \ker(f)$ for all i, j; fixing i, letting j tend to ∞ and recalling that $\ker(f)$ is closed, we see that $v_i \in \ker(f)$. Hence $w = 0$; in other words, $\|\cdot\|_W$ is a norm.

By a similar argument, any Cauchy sequence in W converges: if $\{w_i\}$ is a Cauchy sequence in W, we can choose lifts $\{v_i\}$ of the w_i to V so that $\|v_i - v_{i+1}\| \to 0$ as $i \to \infty$. Since V is complete, the v_i converge to a limit v such that $\|f(v) - w_i\|_W \to 0$ as $i \to \infty$, so $\{w_i\}$ has a limit. We conclude that W W inherits from W the structure of a Banach space.

A lot of stuff you know about Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ carries over to this setting (so I'm not going to bother redoing the classical proofs in these cases; see the references). Typical examples:

- Any finite dimensional vector space over K is a Banach space, and any two norms on it are equivalent.

- A map $f : V \to W$ between Banach spaces is continuous (for the norm topologies) if and only if it is bounded (i.e., there exists $c > 0$ such that $\|f(v)\| \leq c\|v\|$ for all $v \in V$).

- Open mapping theorem: if $f : V \to W$ is a bounded surjective linear map between Banach spaces, then f is an open map (the image of an open subset is open), and the norm topology on W coincides with the quotient topology. More precisely, there exists $c > 0$ such that any $w \in W$ is the image of some $v \in V$ with $\|v\| \leq c\|w\|$. Corollary: any bijective bounded linear map between Banach spaces is an isomorphism. (If anyone wants me to go through the proof of this, let me know and I'll prepare it for next time.)

- Closed graph theorem: the linear map $f : V \to W$ between two Banach spaces is bounded if and only if its graph is closed under the product topology on $V \oplus W$. (Apply the open mapping theorem to the map between W and the quotient of $V \oplus W$ by the graph of f.)

However, the Hahn-Banach theorem extends verbatim *only* to spherically complete fields. For more general fields, one needs an extra restriction on V. We say V is *of countable type* if it contains a countable subset whose linear span is dense in V. (I think such a space in the classical setting is said to be "separable", but that word is sufficiently overburdened in algebraic geometry!)

Lemma 1. *Suppose V is a Banach space of countable type over K. Then for each $R > 1$, there exists an at most countable set $\{e_i\}$ such that each $v \in V$ can be written as a convergent sum $\sum c_i e_i$ with $|c_i| \cdot \|e_i\| \to 0$, and that any such sum satisfies*

$$R^{-1} \max_i \{|c_i| \cdot \|e_i\|\} \leq \|\sum c_i e^i\| \leq \max |c_i| \cdot \|e_i\|.$$

Proof. Exercise, or see [FvdP, Proposition 1.2.1]. □

Theorem 2 (Hahn-Banach). *Let $W \subset V$ be an inclusion of normed spaces, with V complete, and suppose $f : W \to K$ is a bounded K-linear map; that is, there exists $c > 0$ such that $|f(w)| \leq c\|w\|$ for all $w \in W$. Suppose further that* either*:*

(a) V is of countable type, or

(b) K is spherically complete.

Then for any $R > 1$, there exists a K-linear map $g : V \to K$ extending f such that $|g(v)| \leq cR\|v\|$ for all $v \in V$. Moreover, in case (b), we also have this conclusion with $R = 1$.

Proof. For (a), we may assume W is closed because any bounded linear map extends uniquely from W to its closure. In that case, apply Lemma 1 to V/W to produce $d_i \in V$ such that each $v \in V$ has a unique presentation as $w + \sum c_i d_i$ with $w \in W$, $c_i \in K$ and $|c_i| \to 0$. Then extend f by setting $g(w + \sum c_i d_i) = f(w)$. For (b), see Schneider's book; we won't use this part very much. (Again, let me know if you want to see this in detail.) □

Note that the Hahn-Banach theorem *always* fails in general if K is not spherically complete: if L is a spherically complete field containing K, and $D_1, D_2, \dots$ is a decreasing sequence of balls in K with empty intersection, then the identity map $K \to K$ cannot extend to a bounded map $L \to K$, because any element of L in the intersection of the D_i has nowhere to go in K! (Thanks to Damiano Testa for noticing this.)

Exercises

1. Let K be an ultrametric field and let L be a finite extension of K. Show that the absolute value of K extends uniquely to an absolute value on L, and that L is complete if K is.

2. Prove that $\mathbb{C}_p$ (the completion of the algebraic closure of $\mathbb{Q}_p$) is algebraically closed (this is basically Krasner's Lemma), but not spherically complete.

3. Prove that every ultrametric field is contained in a spherically complete field with the same value group and residue field. (Hint: take a bad descending sequence of balls, stick something in it without changing the value group or residue field, then Zornicate.) This implies the equivalence between "spherically complete" and "maximally complete".

4. Prove Lemma 1 (or look it up in [FvdP, Proposition 1.2.1]).

5. Let k be a field, and let $k((t^{\mathbb{Q}}))$ denote the set of formal sums $\sum_{i \in \mathbb{Q}} c_i t^i$, with each $c_i \in k$, whose support (the set of i such that $c_i \neq 0$) is well-ordered (contains no infinite decreasing subsequence). Prove that formal addition and multiplication of these are well-defined, and that they form a field under these operations. These gadgets are variously called *Hahn series* (because Hahn introduced them in 1907), *Mal'cev-Neumann*

series (because Mal'cev and Neumann independently gave vast generalizations), or *generalized power series*. If you really must look this up, see Chapter 13 of Passman's book *The Algebraic Structure of Group Rings*.

6. Prove that $k((t^{\mathbb{Q}}))$ is spherically closed. Deduce that if k is algebraically closed, then so is $k((t^{\mathbb{Q}}))$.

7. (from Bjorn Poonen's undergraduate thesis) Give an explicit construction of a spherically complete field containing $\mathbb{C}_p$. (Hint: you want to do something like making $k((t^{\mathbb{Q}}))$, but starting from $\mathbb{Q}_p^{\mathrm{unr}}$ and writing down "generalized power series in p". You can make that make sense by quotienting an appropriate ring of things looking like generalized power series by the ideal $p - t$. Or see my paper "Power series and p-adic algebraic closures".)