18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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A little p-adic functional analysis (part 2 of 2)

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Orthonormal bases

An *orthogonal basis* of a Banach space V is a subset $\{e_i\}_{i\in I}$ of V with the property that each $m \in M$ has a unique representation as a convergent sum $\sum_{i\in I} c_i e_i$, and if one always has $\|m\| = \max_{i\in I}\{\|c_i e_i\|\}$. (Note that "convergent" only makes sense if I is at most countable; but note also that there is no distinction between "convergent" and "absolutely convergent" in the nonarchimedean setting!) The basis is *orthonormal* if $\|e_i\| = 1$ for each i. Lemma 1 from last time says that the norm on a Banach space of countable type can be approximated by equivalent norms which admit orthogonal bases; see below for an example.

Banach algebras

A *Banach algebra (over K)* is a Banach space A over K which is also a commutative K-algebra, and which satsifies the following additional restrictions.

(a) $\|1\| = 1$.

(b) for $x, y \in A$, $\|xy\| \leq \|x\| \cdot \|y\|$.

A *Banach module over A* is a Banach space M equipped with an A-module structure, such that $\|am\| \leq \|a\| \cdot \|m\|$ for $a \in A$ and $m \in M$.

Here's an example where life turns out to be easier than in the real/complex case [FvdP, Lemma 1.2.3]. I'll start next time with an example of this.

Lemma 1. *Let A be a Banach algebra over K which is noetherian as a ring. Let M be a Banach module over A which is module-finite over A. Then any A-submodule of M is closed.*

Proof. Let N be a submodule of M and let $\tilde{N}$ be the closure of N. Since A is noetherian, $\tilde{N}$ is module-finite over A; choose generators $e_1, \dots, e_n$ of $\tilde{N}$ over A. Consider the A-module homomorphism $A^n \to \tilde{N}$ defined by $(a_1, \dots, a_n) \mapsto \sum a_i e_i$, where A^n is equipped with the norm $\|(a_1, \dots, a_n)\| = \max_i\{\|a_i\|\}$. By the open mapping theorem, there exists $c \in (0, 1)$ such that each $x \in \tilde{N}$ can be written as $\sum a_i e_i$ with $c \max_i\{\|a_i\|\} \leq \|x\|$. Choose $f_1, \dots, f_n \in N$ with $\|e_i - f_i\| \leq c^2$; we show that $f_1, \dots, f_n$ also generate $\tilde{N}$, which implies that $N = \tilde{N}$.

Given $x \in \tilde{N}$, define the sequence $x_0, x_1, \dots$ as follows. Set $x_0 = x$; given x_j, write $x_j = \sum_i a_{j,i} e_i$ with $c \max_i\{\|a_{j,i}\|\} \leq \|x_j\|$, and put

$$x_{j+1} = \sum a_{j,i}(e_i - f_i),$$

so that $\|x_{j+1}\| \leq c\|x_j\|$. This means $x_j \to 0$ and so for each i, the series $\sum_j a_{j,i}$ converges to a limit a_i satisfying $x = \sum a_i f_i$. Thus $N = \tilde{N}$, as desired. $\square$

Tensor products

The tensor product of two Banach spaces V and W is not complete, so instead we will typically work with the *completed tensor product* $V \widehat{\otimes} W$, which as the name suggests is just the completion of the ordinary tensor product as K-vector spaces.

The completed tensor product is also a Banach space, but this takes a bit of work to check. One gets a seminorm on $V \otimes W$ from the formula

$$\|x\| = \inf\{\max_i\{\|v_i\| \cdot \|w_i\|\}\},$$

the infimum taken over all presentations $x = \sum_{i=1}^m v_i \otimes w_i$, and this extends to the completion. To check that it's a norm, one needs to check that if $\{x_j\}_{j=1}^\infty$ is a sequence of elements of $V \otimes W$, with $x_j = \sum_{j=1}^{m_j} v_{ij} \otimes w_{ij}$, and $\max_i\{\|v_{ij}\| \cdot \|w_{ij}\|\} \to 0$ as $j \to \infty$, then $x_j \to 0$ in $V \otimes W$. We may check this after replacing V and W by the span of the v_{ij} and w_{ij}, respectively, dropping us into the countable type case. Then writing everything in terms of an orthogonal basis of an equivalent norm (using Lemma 1 from last time) yields the claim.

The completed tensor product is exact in the following sense. We say a sequence

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

of maps between Banach spaces is *exact* if it is exact in the usual sense, and also f and g are isometric. (For f, this means the norm on M_1 is the restriction from M_2; for g, the norm on M_3 is the quotient norm from M_2.) Then for any Banach space N, the sequence

$$0 \to M_1 \widehat{\otimes} N \xrightarrow{f} M_2 \widehat{\otimes} N \xrightarrow{g} M_3 \widehat{\otimes} N \to 0$$

is exact.

Warning: if A is a Banach algebra, the product seminorm on a tensor product $M \otimes_A N$ of Banach modules M and N over A may not be a norm! But see exercises for an important case where this is okay.

Exercises

1. (Leftover from last time) Prove that $\cup_{n=1}^\infty k((t^{1/n}))$ is algebraically closed for $k = \mathbb{C}$, but not for any field k of positive characteristic. (Hint: look at $P(x) = x^p - x - t^{-1}$.)

2. Suppose K is a complete discretely valued (ultrametric) field. Let M be a Banach space over K such that $\|m\| \in |K|$ for each $m \in M$. Put $\mathfrak{o}_M = \{m \in M : \|m\| \leq 1\}$ and $\overline{M} = \mathfrak{o}_M \otimes_{\mathfrak{o}_K} k$. Prove that a subset of M forms an orthonormal basis if and only if its image in $\overline{M}$ is a basis of $\overline{M}$ as a k-vector space. (Hint: see [FvdP, Lemma 1.2.2].)

3. Let A be a Banach algebra which is noetherian as a ring, and let M and N be Banach modules over A which are module-finite over A. Show that the product seminorm

$$\|x\| = \inf\{\max_i\{\|m_i\| \cdot \|n_i\| : x = \sum m_i \otimes n_i\}$$

is a norm on $M \otimes N$, and that $M \otimes N$ is complete for this norm. (Hint: choose finite presentations of M and N and reduce to the case of free modules.)