18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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Subsets of the projective line, part 1

Before proceeding with more general theory, let's look at some concrete examples of affinoid spaces. We'll also look at some spaces that by all rights should be admissible in rigid analytic geometry but which are not affinoid.

Reference: [FvdP, Chapter 2].

Affinoid subsets of $\mathbb{P}^1$

As usual, $\mathbb{P}^1$ will denote the projective line over $\operatorname{Spec} \mathbb{Z}$, which you may then base-change to any base scheme. Here I'm going to work with a complete ultrametric field K. [FvdP] work with the set of K-valued points $\mathbb{P}^1(K)$, but this is not really the right way to look at this; I will instead define the set $\mathbb{P}$ on which I work to be the set of *closed points* of the scheme $\mathbb{P}^1_K$. However, when it's convenient to do so, I will represent a closed point by a point of $\mathbb{P}^1(K^{\mathrm{alg}})$ contained in it; I hope this won't cause too much confusion.

Also, let Γ be the divisible closure in $\mathbb{R}_{>0}$ of the group $|K^*|$; that is, $\Gamma = |(K^{\mathrm{alg}})^*|$.

A *closed disc* in $\mathbb{P}$ is a subset of one of the forms

$$\{x \in K^{\mathrm{alg}} : |x - a| \leq r\}$$

or

$$\{x \in K^{\mathrm{alg}} : |x - a| \geq r\} \cup \{\infty\}$$

for some $a \in K$ and $r \in \Gamma$. An *open disc* is the same thing but with strict inequalities.

A *connected affinoid subset* of $\mathbb{P}$ is the complement of a nonempty finite union of open discs. An *affinoid subset* of $\mathbb{P}$ is a finite union of connected affinoid subsets.

We want to say that the affinoid subsets are "affinoid spaces" in some natural sense, but we can't yet say that because we haven't abstractly defined an affinoid space. The best we can do for now is "naturally" identify them with the maximal spectra of certain affinoid algebras.

Example: the closed disc $|x| \leq 1$ is just $\operatorname{Max} K\langle x \rangle$. Keep this example firmly in mind as we go along!

0.1 Holomorphic functions

Let F be an affinoid subset of $\mathbb{P}$, and let $R(F) \subset K(x)$ be the set of rational functions whose poles do not lie in F. Let $\|\cdot\|_F$ denote the supremum norm on $R(F)$:

$$\|f\|_F = \sup_{a \in F}\{|f(a)|\}.$$

Let A_F be the completion of $R(F)$ with respect to $\|\cdot\|_F$.

Theorem 1.

(a) *The ring A_F is a reduced affinoid algebra.*

(b) *The natural map $F \to \operatorname{Max} A_F$ is a bijection.*

(c) *Under the map in (b), the supremum norm corresponds to the spectral norm on A_F.*

Proof. We can decompose F uniquely as a disjoint union of connected affinoid subsets (see exercises), so it suffices to do the case where F is connected. Also, by performing a fractional linear transformation, I can reduce to the case where F is contained in the closed unit disc (so in particular $\infty \notin F$).

Suppose F is defined by the equations

$$|x - a_i| \leq r_i \quad (i = 1, \dots, m), \qquad |x - b_j| \geq s_j \quad (j = 1, \dots, n).$$

Choose positive integers $e_1, \dots, e_m, f_1, \dots, f_m$ and elements $\rho_1, \dots, \rho_m, \sigma_1, \dots, \sigma_n \in K$ with $|\rho_i| = r_i^{e_i}$ and $|\sigma_j| = s_j^{f_j}$. Let I be the ideal of $K\langle x, y_1, \dots, y_m, z_1, \dots, z_n\rangle$ generated by

$$\rho_i y_i - (x - a_i)^{e_i} \quad (i = 1, \dots, m), \qquad \sigma_j - (x - b_j)^{f_i} z_j \quad (j = 1, \dots, n).$$

Then one checks (using the hypothesis on the placement of F) that A_F is the maximal reduced quotient of

$$K\langle x, y_1, \dots, y_m, z_1, \dots, z_n\rangle / I$$

and that the other properties hold. Actually the way to do this is in reverse: first observe (by the Nullstellensatz) that $\operatorname{Max} A_F = F$, so the spectral norm coincides with the supremum norm, then use the theorem from last time that A_F is complete for the supremum norm. $\square$

Exercises

1. (from [FvdP, Exercise 2.1.1]) Prove that every affinoid subset can be written uniquely as a disjoint union of connected affinoid subsets.

2. Suppose that in the proof of Theorem 1, the e_i and f_j are chosen to be as small as possible. Prove that the quotient

$$K\langle x, y_1, \dots, y_m, z_1, \dots, z_n\rangle / I$$

is already reduced, so it is actually equal to A_F.

3. Suppose that K is spherically complete. Give an explicit example of an affinoid algebra A which is an integral domain, but whose spectral seminorm is not fully multiplicative. (Hint: consider power series in x and x^{-1} which converge on a suitable annulus; your motivation should functions on a complex annulus which have their suprema on opposite boundary components. We'll look more at this geometric situation in the next few lectures.)