18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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Subsets of the projective line (and G-topologies), part 2

Reference: [FvdP, Chapter 2]. Note: I'm going to go through the examples a bit quickly, because I think it's easy enough for you to fill in the details; or see [FvdP]. See also [BGR, 9.1] for G-topologies.

Affinoid subsets of $\mathbb{P}^1$ revisited

Let $F \subset \mathbb{P}^1$ be an affinoid subset of $\mathbb{P}^1$. Last time, we showed that the ring A_F, the completion for the supremum norm of the ring of rational functions with poles outside F, is an affinoid algebra whose maximal ideals are precisely the points of F.

It will sometimes be useful to cover a connected affinoid subset with "standard" pieces. Namely, let F be a connected affinoid subset of $\mathbb{P}$ containing ∞, and write F as the complement of the union of the open discs

$$D_i = \{a \in \mathbb{P} : |a - a_i| < r_i\} \qquad (a_i \in K, r_i \in \Gamma).$$

Put $F_i = \mathbb{P} \setminus D_i$. Put $A = A_F$ and $A_i = A_{F_i}$. If F doesn't contain ∞, you can do likewise but with one "everted" open disc that contains ∞.

Proposition 1. *Let F be a connected affinoid subset of $\mathbb{P}$, and pick some $a \in \mathbb{P}^1(K) \setminus F$. Then any element of A_F can be written uniquely as a rational function, with all zeroes inside F and all poles at a, times a unit of A_F.*

Proof. First note that if F is the closed unit disc and $a = \infty$, then we already know (by Weierstrass preparation) that every element of $A_F = K\langle x \rangle$ is equal to a polynomial in x times a unit. Moreover, we can factor that polynomial into a part whose roots have norm ≤ 1 and a part whose roots have norm > 1, and the latter is invertible in A_F. This gives existence of the desired factorization; uniqueness follows because every point in the closed unit disc really does give rise to a maximal ideal of A_F.

Now consider the general statement. We first check that each $f \in A_F$ has only finitely many zeroes. It suffices to check this on each F_i, and also it doesn't hurt to replace K by a finite extension. But note that after tensoring with a finite extension of K (whose image under $|\cdot|$ contains the radius of the disc D_i), A_{F_i} becomes isomorphic to a Tate algebra, which we already looked at above.

There exists a unique rational function g, with all zeroes inside F and all poles at a, whose zeroes are precisely those of f with the same multiplicities, and this rational function visibly belongs to f. Moreover, f is divisible by g in the localization of A_F at each maximal ideal, so $f/g \in A_F$; likewise, $g/f \in A_F$. This gives existence of the desired factorization; uniqueness again follows because each point of F really corresponds to a maximal ideal of A_F. $\square$

Corollary 2. *If F is connected, then A_F is a principal ideal domain.*

Another interesting class of examples are the annuli

$$F = \{x \in \mathbb{P} : r_1 \leq |x| \leq r_2\},$$

on which analytic functions are given by Laurent series $\sum_{n=-\infty}^{\infty} c_n x^n$, with $c_n \in K$, which converge for $r_1 \leq |x| \leq r_2$. (Note that if one of r_1 or r_2 is in $\Gamma = |(K^{\text{alg}})^*|$ but not in $|K^*|$, you have to check this radius of convergence using points in $\mathbb{P}$, not just K-rational points.) If $r_1 = r_2 = 1$, you get what [FvdP] calls a *ring domain*; see [FvdP,Example 2.2.5].

G-topologies

One has a natural topology on $\mathbb{P}$ induced by the metric topology on K, but this topology is much too fine to be of any use in doing analytic geometry. In fact the same is true of $\operatorname{Max} A$ of any affinoid space. To view these objects as locally ringed spaces in a sensible fashion, we need a better topology; unfortunately, this will have to be a Grothendieck topology, but one of a particularly simple form.

Let X be a set. A *G-topology* on X consists of the following data:

(i) a family of subsets of X containing $\emptyset$ and F, and closed under finite intersections (the *admissible subsets*);

(ii) for each admissible subset, a set of (set-theoretic) coverings of U by admissible subsets (the *admissible coverings*);

subject to the following conditions.

(a) The covering $\{U\}$ of an admissible subset by itself is always admissible.

(b) If U, V are admissible subsets with $V \subset U$ and $\{U_i\}_{i \in I}$ is an admissible covering of U, then $\{U_i \cap V\}_{i \in I}$ is an admissible covering of V.

(c) If U is an admissible subset, $\{U_i\}_{i \in I}$ is an admissible covering, and we are given an admissible covering of each U_i, then the union of these coverings is an admissible covering of U.

We also call admissible subsets *admissible open subsets*, or even *admissible opens*. If we give the topology a name T, we will speak of T-admissible opens and coverings, or even just T-opens and T-coverings.

If you've seen Grothendieck topologies before, you should be on familiar territory. If not, keep in mind that the point of this definition is to isolate, out of the usual concept of a topology, the bare minimum needed to work with sheaves. Namely, a *presheaf* on a G-topology is a contravariant functor $\mathcal{H}$ from the category of admissible subsets (with morphisms being inclusions) to sets (or whatever other objects you have in mind). That is, for each inclusion $U \subseteq V$ of admissible opens, you get a restriction map $\operatorname{Res}_{V,U} : \mathcal{H}(V) \to \mathcal{H}(U)$, and these compose as you expect. (so in particular $\operatorname{Res}_{U,U}$ is the identity). The presheaf is a *sheaf* if the sheaf axiom is satisfied: whenever $\{U_i\}_{i \in I}$ is an admissible covering of an admissible open U, specifying an $f_i \in \mathcal{H}(U_i)$ such that $\mathrm{Res}_{U_i,U_i \cap U_j}(f_i) = \mathrm{Res}_{U_j,U_i \cap U_j}(f_j)$ for all i, j uniquely specifies an $f \in \mathcal{H}(U)$ such that $\mathrm{Res}_{U,U_i}(f) = f_i$. Basically everything you know about sheaves carries over to this context: there is a "sheafification" functor, one can make Čech complexes, and so on.

Since all we care about is the sheaf theory on a topology, we want to consider two topologies with the same sheaf theory to be "equivalent". To wit, we say a G-topology T' is *finer* than another G-topology T on the same set if every T-open is T'-open and every T-covering is a T'-covering. We say T' is *slightly finer* than T if T' is finer than T, and also:

(a) every T'-open has a T'-covering by T-opens;

(b) every T'-covering of a T-open can be refined to a T-covering.

The categories of (pre)sheaves on T and T' are the same, and the computation of Čech cohomology is the same; this is all easy but boring to show, so see [BGR,Chapter 9] for details. (In more precise abstract nonsense terms, these two topologies determine the same "topos".)

A map between sets equipped with G-topologies is *continuous* if every admissible open pulls back to an admissible open, and every admissible covering pulls back to an admissible covering.

Next time, we'll construct some G-topologies on $\mathbb{P}$ and play with them a bit.

Exercises

1. Let $I \subset [0, \infty)$ be an interval whose left endpoint is either 0 or lies in Γ, and whose right endpoint lies in Γ, and put

$$F = \{x \in \mathbb{P} : |x| \in I\}$$

(so that F is either a closed disc or an annulus). Let M be an invertible $n \times n$ matrix over A_F. Then there exists an invertible $n \times n$ matrix U over $K[t]$ in case $0 \in I$, or $K[t, t^{-1}]$ in case $0 \notin I$, such that $|MU - I_n|_{A_F,\mathrm{spec}} < 1$. (Hint: perform "approximate Gaussian elimination". If you get stuck, see Section 3.2 of my preprint "Semistable reduction II", on my web site.)

2. (Mittag-Leffler decompositions; [FvdP, Proposition 2.2.6]) Let F be a connected affinoid subset of $\mathbb{P}$ containing ∞, and write F as the complement of the union of the open discs

$$D_i = \{a \in \mathbb{P} : |a - a_i| < r_i\} \qquad (a_i \in K, r_i \in \Gamma).$$

Put $F_i = \mathbb{P} \setminus D_i$. Put $A = A_F$ and $A_i = A_{F_i}$, and put

$$A_+ = \{f \in A : f(\infty) = 0\}, \qquad A_{i,+} = \{f \in A_i : f(\infty) = 0\}.$$

Prove that $A_+ = \oplus_i A_{i,+}$, and that for any elements $f_i \in A_{i,+}$, one has

$$\|f_i\|_F = \|f_i\|_{F_i}, \qquad \|\sum_i f_i\|_F = \max_i\{\|f_i\|_F\}.$$

3. Given a G-topology T, prove there is a unique finest G-topology T' on the same set among those which are slightly finer than T. (Hint: see [BGR, 9.1].)