18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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Rigid analytic spaces (at last!)

We are now ready to talk about rigid analytic spaces in earnest. I'll give the definition and then some examples; we may discuss some of these examples in more detail, as interest dictates.

References: [FvdP, Chapter 4] and [BGR, Chapter 9]. Additional references are given throughout the text.

Locally G-ringed spaces and rigid spaces

A *locally G-ringed space* is a set X with a G-topology and a sheaf of rings $\mathcal{O}_X$ whose stalks at each $x \in X$ are local rings. A morphism between two such gadgets $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ (i.e., one pulling admissible opens/coverings back to admissible opens/coverings) and a sheaf-of-rings homomorphism $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ such that the induced homomorphisms on stalks are local homomorphisms. As usual, the pushforward f_* of a $\mathcal{O}$-module is just the sheaf-theoretic direct image and the pullback is the sheaf-theoretic inverse image tensored up to $\mathcal{O}_X$ using the homomorphism $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. As pointed out last time, any affinoid space is a locally G-ringed space: the stalk at a point coincides with the local ring of the affinoid algebra at the corresponding maximal ideal.

A *(very weak, weak, somewhat weak, strong) affinoid space* is a locally G-ringed space of the form $\operatorname{Max} A$, for some affinoid algebra A, equipped with the corresponding G-topology. Note that homomorphisms between affinoid algebras give rise to morphisms of the corresponding affinoid spaces for any of the four types of G-topologies.

Lemma 1. *The contravariant functor from affinoid algebras to affinoid spaces (viewed as a subcategory of the locally G-ringed spaces) is an equivalence for any of the four types of G-topologies, with quasi-inverse given by the global sections functor.*

Proof. The nontrivial assertion here is that there can be at most one morphism of locally G-ringed spaces with a given action on global sections. The action on points is uniquely determined because the induced homomorphisms on stalks are local; the sheaf map is uniquely determined by the universal property defining affinoid subdomains. For more details, see [BGR, Proposition 9.3.1/1]. □

A *(very weak, weak, somewhat weak, strong) rigid analytic space* (over K) is a locally G-ringed space X for which there exists an admissible covering $\{U_i\}_{i \in I}$ of X with the following properties.

(a) Each U_i is a (very weak, weak, somewhat weak, strong) affinoid space.

(b) A subset U of X is admissible if and only if $U \cap U_i$ is admissible for each $i \in I$.

Note that the concepts become more expansive as you make the topology finer; the term "rigid analytic space" without qualification usually means a strong rigid analytic space.

A *coherent* (resp. *coherent locally free*) sheaf on a rigid analytic space is one which on the elements of some admissible affinoid covering looks like the sheaf associated to a finitely generated (resp. finite free) module over the structure sheaf. Of course, on a general space, a coherent sheaf need not be generated by its global sections.

A *closed analytic subspace* of a rigid space X is a subspace which on the elements of some admissible affinoid covering looks like the zero locus of some ideal. (This is what you might think of as being analogous to a "closed subscheme".)

Example: generic fibres

A nice set of relatively simple examples of rigid spaces come from Raynaud's "generic fibre" construction.

Let $P = \operatorname{Spf} A$ be an affine formal scheme of finite type over $\mathfrak{o}_K$, that is, A is an $\mathfrak{o}_K$-algebra complete for the ideal $\mathfrak{m}_K A$ and topologically finitely generated over $\mathfrak{o}_K$. Then $A_K = A \otimes_{\mathfrak{o}_K} K$ is an affinoid algebra; we call the corresponding affinoid space $\operatorname{Max} A_K$ the *generic fibre* of P. If P is not affine, we can construct the generic fibre by glueing this construction.

The points of the generic fibre correspond to subschemes of P which are integral and finite flat over $\mathfrak{o}_K$. In particular, there is a specialization map $\mathrm{spe} : A_K \to P_k$ (where $P_k = P \otimes_{\mathfrak{o}_K} k$, and k is as always the residue field of K) taking one of these points to its special fibre. Note that the generic fibre consists of the space *plus* the specialization map; the space itself isn't enough to recover P.

If P is projective, then the generic fibre can be identified with the closed points of the usual generic fibre. In this case, one has a form of the "GAGA principle": any coherent sheaf on the analytic generic fibre is algebraic, and the analytic (Čech) and algebraic cohomologies coincide. This is one of Kiehl's theorems, on which more at a later date.

Incidentally, one sometimes wants to form the "generic fibre" of things which are not topologically finitely generated over $\mathfrak{o}_K$, like $\mathfrak{o}_K[\![t]\!]$, whose generic fibre should be the open unit disc over K. One has to be a bit careful: the ring $\mathfrak{o}_K[\![t]\!]$ is not a valuation ring, because there are series $\sum c_n t^n$ with $|c_n| < 1$ for all n but $|c_n| \to 1$ as $n \to \infty$. In fact, I don't know a good general construction to put here; any suggestions?

Example: the Tate curve, for real this time

Let X be a rigid space, and let Γ be a group acting on X. We say the action of Γ is *discontinuous* if X admits an admissible covering $\{U_i\}_{i \in I}$ by affinoids such that for each i, the set of $\gamma \in \Gamma$ such that $X_i^\gamma \cap X_i \neq \emptyset$ is finite. If this set only ever consists of the identity element of Γ, the action is *free*.

Let $\mathbb{G}_{m,K}$ be the rigid analytic multiplicative group over K; if you like, you can think of it as the result of removing 0 and ∞ from the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$ (which is just the space

$\mathbb{P}$ we discussed before). For $q \in K$ with $|q| > 1$, the action of $\mathbb{Z}$ on $\mathbb{G}_{m,K}$ in which n acts by multiplication by q^n is free, since we can cover $\mathbb{G}_{m,K}$ admissibly with the affinoids

$$|q|^{n/2} \leq |x| \leq |q|^{(n+1)/2}$$

and no one of these meets its image under the action of a nonzero $n \in \mathbb{Z}$.

For more, see [FvdP, Section 5.1].

Example: Mumford curves

The group $\mathrm{PGL}_2(K)$ acts on $\mathbb{P}$, the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$. For Γ a subgroup of $\mathrm{PGL}_2(K)$, let $\mathcal{L} \subseteq \mathbb{P}$ be the set of limit points of Γ (that is, the set of $a \in \mathbb{P}$ for which there exists a point $b \in \mathbb{P}$ and a sequence $\gamma_1, \gamma_2, \ldots$ of distinct elements of Γ with $b^{\gamma_n} \to a$ as $n \to \infty$). We say Γ is *discontinuous* if $\mathcal{L} \neq \mathbb{P}$, and the topological closure of any orbit is compact.

A *Schottky group* is a finitely generated nontrivial discontinuous group Γ with no nontrivial finite subgroup. For Γ a Schottky group, let Ω be the complement of $\mathcal{L}$ in $\mathbb{P}$. Then we may form the quotient Ω/Γ (since the action is discontinuous in the sense of the previous section), and—a la peanut butter sandwiches![1]—the result is the analytification of a smooth projective curve, called a *Mumford curve*. The group Γ is necessarily free of some finite number g of generators, and g is also the genus of the curve. There is more combinatorial data hidden in this description, on which Γ is acting (including information about the reduction type of the curve), and the whole picture is tied up with the theory of Shimura varieties, and with stable reduction of curves.

Related example: the *Drinfel'd upper half-space* of dimension n over K is the subspace of the analytified projective space $\mathbb{P}^n_K$ minus the union of all K-rational hyperplanes. The group $\mathrm{PGL}_n(K)$ acts on this space, and one gets a lot of interesting spaces by forming quotients by discrete subgroups. This sort of business goes under the heading of "p-adic uniformization".

For more, see [FvdP, Section 5.4].

G-topologies and ordinary topologies

I wanted to mention a result that compares Čech cohomology and sheaf cohomology on some rigid spaces, but first I need to fix the fact that affinoid spaces do not have enough points. This will involve Before giving the statement, I need some "extra points" on an affinoid space, or more generally on any G-topological space. Given a space X equipped with a G-topology, a *G-filter* (or simply "filter") on X is a collection $\mathcal{F}$ of admissible subsets with the following properties.

(a) $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.

(b) If $U_1, U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$.

[1] The "abracadabra" phrase of The Amazing Mumford, the magician Muppet on *Sesame Street*. I have been unable to confirm reports that this Muppet is actually named after Mumford the mathematician.

(c) If $U_1 \subseteq U_2$ and $U_1 \in \mathcal{F}$, then $U_2 \in \mathcal{F}$.

A *prime filter* is a filter $\mathcal{F}$ also satisfying

(d) if $U \in \mathcal{F}$ and $\{U_i\}_{i \in I}$ is an admissible covering of U, then $U_i \in \mathcal{F}$ for some $i \in I$.

A *maximal filter* (or *ultrafilter*) is a filter $\mathcal{F}$ which is maximal under inclusion; such a filter is clearly also prime. For each $x \in X$, the set of admissibles containing x is a maximal filter.

Let $\mathcal{P}(X)$ and $\mathcal{M}(X)$ denote the sets of prime and maximal filters, respectively, on X, and likewise for any admissible open U of X (that is, $\mathcal{P}(U)$ consists of prime filters of X in which U appears). Equip $\mathcal{P}(X)$ with the ordinary topology generated by the $\mathcal{P}(U)$; then there is a natural morphism of sites $\sigma : X \to \mathcal{P}(X)$, and it turns out that the functors σ_* and σ^* are equivalences between the categories of abelian sheaves on X and on $\mathcal{P}(X)$ [FvdP, Theorem 7.1.2].

The topological space $\mathcal{P}(X)$ is typically pretty unwieldy. For rigid analytic spaces, Berkovich theory gives a way to recover all of the information in $\mathcal{P}(X)$ by working on $\mathcal{M}(X)$, which is still a bigger space than X itself but is small enough to be more wieldy (and is more closely analogous to spaces you see in ordinary analysis). We'll introduce this perspective sometime later in the term.

Čech versus sheaf cohomology

Here's a precise comparison statement between Čech and sheaf cohomology (it may not be optimal, but almost surely some sort of finiteness hypothesis is necessary). See van der Put, Cohomology of affinoid spaces, *Comp. Math.* **45** (1982), 165–198, Proposition 1.4.4.

Proposition 2. *Let X be a strong rigid analytic space with an at most countable admissible covering $\{U_i\}_{i \in I}$ such that each U_i is an affinoid space, and each $U_i \cap U_j$ is an affinoid subspace of U_i. Then Čech cohomology computes sheaf cohomology for any abelian sheaf on X.*

Note that I said any abelian sheaf, not necessarily a coherent sheaf.

Sketch of proof. Put $\mathcal{P}(X) = \cup_i \mathcal{P}(U_i)$. Given a sheaf $\mathcal{F}$, pick one injective abelian group $\mathcal{G}_p$ containing $\mathcal{F}_p$ for each $p \in \mathcal{P}(X)$, and define the presheaf $\mathcal{G}_p$ on $\mathcal{P}(X)$ by $\mathcal{G}(U) = \prod_{p \in U} \mathcal{G}_p$. Then $\mathcal{G}$ is a sheaf and its stalk at p is $\mathcal{G}_p$; in particular, $\mathcal{G}$ is injective. Let $\mathcal{H}$ be the presheaf cokernel of the injection $\mathcal{F} \to \mathcal{G}$; then $\mathcal{H}^+$ is a sheaf, and coincides with the sheafification of $\mathcal{H}$.

Define $\mathcal{K}$ as the presheaf cokernel of $\mathcal{H} \to \mathcal{H}^+$, so that $\mathcal{K}^+ = 0$. At this point one must verify that for X as chosen, the fact that $\mathcal{K}^+ = 0$ implies that $\check{H}^i(X, \mathcal{K}) = 0$ for all i. This, despite being more or less formal, is the heart of the matter; see Lemma 1.4.5 of the aforecited paper. (The point of the hypothesis on X is that any cover can be refined to a countable cover by affinoid subspaces.)

Given that $\check{H}^i(X, \mathcal{K}) = 0$ for all i, the Čech cohomologies of $\mathcal{H}$ and $\mathcal{H}^+$ must coincide. We thus have an exact sequence

$$0 \to \check{H}^0(X, \mathcal{F}) \to \check{H}^0(X, \mathcal{G}) \to \check{H}^0(X, \mathcal{H}^+) \to \check{H}^1(X, \mathcal{F}) \to 0$$

and isomorphisms $\check{H}^i(X, \mathcal{H}^+) \to \check{H}^{i+1}(X, \mathcal{F})$. Comparing the first sequence with the analogous sequence in sheaf cohomology, you get $\check{H}^1(X, \mathcal{F}) \cong H^1(X, \mathcal{F})$ for all $\mathcal{F}$. Now proceed by induction and "dimension shifting": given $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ for all $\mathcal{F}$, apply this isomorphism with $\mathcal{F}$ replaced by $\mathcal{H}^+$. $\square$

Exercises

1. (Rigid Hartogs' lemma) Let X be the rigid space obtained from $\operatorname{Max} K\langle x_1, \dots, x_n\rangle$, for some $n \geq 2$, by removing the point $(0, \dots, 0)$. Prove that $\mathcal{O}(X) = K\langle x_1, \dots, x_n\rangle$.

2. Suppose that K is spherically complete. Prove that every coherent locally free sheaf on the open annulus $\{x \in \mathbb{P} : r_1 < |x| < r_2\}$ is free. (The rank 1 case is more or less an exercise I gave earlier.) If you get stuck, see my preprint "Semistable reduction... II" on my web site. Warning: a general coherent sheaf on an open annulus need not be generated by global sections!

Problems

These aren't listed as exercises because I don't know how to do them!

1. (from Richard Taylor) Let X be the rigid space obtained from $\operatorname{Max} K\langle x_1, \dots, x_n\rangle$, for some $n \geq 2$, by removing *all* of the K-rational points. Is $\mathcal{O}(X) = K\langle x_1, \dots, x_n\rangle$? How about if you remove all K-rational planes of codimension at least 2?

2. Let X be an affinoid space, let U be a connected (in the G-topological sense) affinoid subspace, and choose $f \in \mathcal{O}(U)$. Let Y be the union of all connected affinoid subspaces V containing U for which there exists $g \in \mathcal{O}(U)$ with restriction f. (I'm trying to think of Y as the "domain of definition" of f.) Does there exist $g \in \mathcal{O}(Y)$ extending f? If so, is g unique? Also, what else can you say about Y (e.g., can one have $Y = U$?)

3. What is the "right" level of generality for the assertion that Čech cohomology computes sheaf cohomology on a rigid analytic space?