18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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More on affinoid algebras

Addenda on the spectral seminorm

A norm on a Banach algebra A is *power-multiplicative* if $\|f^n\| = \|f\|^n$ for any $f \in A$ and any positive integer n. Our proof that the Gauss norm on T_n has a topological characterization adapts to show that for any affinoid algebra A, there is at most one power-multiplicative Banach norm on A. We now know that if a power-multiplicative norm exists, it must be the spectral seminorm; hence such a norm exists if and only if A is reduced.

In fact, the spectral seminorm is "minimal" in the following sense [BGR, Corollary 3.8.2/2].

Proposition 1. *Let A be an affinoid algebra with norm $\|\cdot\|$. Then for all $f \in A$, $\|f\|_{\mathrm{spec}} \leq \|f\|$. In particular, $|f(x)| \leq \|f\|$ for any $x \in \operatorname{Max} A$.*

Proof. Apply the formula
$$\|f\|_{\mathrm{spec}} = \lim_{n\to\infty} \|f^n\|^{1/n}$$
and note that $\|f^n\| \leq \|f\|^n$ because $\|\cdot\|$ is a Banach algebra norm. □

This yields the following characterization of nilpotent elements, in the vein of our characterization of power-bounded elements [BGR, Proposition 6.2.3/2].

Proposition 2. *For A an affinoid algebra and $f \in A$, the following statements are equivalent:*

(a) f is topologically nilpotent (i.e., $\{f^n\}$ is a null sequence in A);

(b) $|f(x)| < 1$ for all $x \in \operatorname{Max} A$;

(c) $\|f\|_{\mathrm{spec}} < 1$.

Proof. The equivalence of (b) and (c) follows from the maximum modulus principle, and (a) implies (c) by the previous proposition. Given (c), choose $c \in K$ and $m \in \mathbb{N}$ such that $|c| > 1$ but $\|cf^m\|_{\mathrm{spec}} \leq 1$. Then cf^m is power-bounded (from last time), so $c^{-1}(cf^m) = f^m$ is topologically nilpotent, as then is f. Thus (c) implies (a), and we are done. □

Spectral norms are Banach norms

We now know that the spectral seminorm on a reduced affinoid algebra is a norm. However, more than that is true: it is a Banach norm. (This proof is from [FvdP, Theorem 3.4.9]; the proof in [BGR, Theorem 6.2.4/1] is a bit more intricate.)

Lemma 3. *Let $A \hookrightarrow B$ be an inclusion of affinoid algebras. Then the spectral seminorm on B restricts to the spectral seminorm on A.*

Proof. Choose a Banach norm $\|\cdot\|$ on B; it then restricts to a Banach norm on A, and applying the formula $\|f\|_{\mathrm{spec}} = \lim_{n\to\infty} \|f^n\|^{1/n}$ gives us the claim. $\square$

Theorem 4. *Let A be a reduced affinoid algebra. Then A is complete under $\|\cdot\|_{\mathrm{spec}}$. In particular, every Banach algebra norm on A is equivalent to the spectral norm.*

Proof. The last sentence will follow from what we showed earlier: any two Banach algebra norms on an affinoid algebra are equivalent. So we focus on showing that A is complete.

We first reduce to the case where A is an integral domain. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal primes of A. Choose a Banach norm $\|\cdot\|_A$ on A, put $A_i = A/\mathfrak{p}_i$ (which is an integral domain), and equip each A_i with the quotient norm induced by $\|\cdot\|_A$. Then $A_1 \oplus \cdots \oplus A_m$ becomes a finitely generated Banach module over A under the max norm

$$(a_1, \dots, a_m) = \max_i\{\|a_i\|\}.$$

Let $i : A \to A_1 \oplus \cdots \oplus A_m$ be the canonical injection; then $i(A)$ is an A-submodule of $A_1 \oplus \cdots \oplus A_m$. By the lemma from "$p$-adic functional analysis 2", $i(A)$ is closed in $A_1 \oplus \cdots \oplus A_m$, and i is an isomorphism onto its image by the open mapping theorem. The map i is isometric for the spectral norms, so proving that the spectral norm on each A_i is equivalent to the quotient norm proves that the spectral norm on A is equivalent to $\|\cdot\|_A$.

From now on, we assume A is an integral domain. If B is a reduced affinoid algebra containing A, showing that B is complete under its spectral seminorm implies that A is complete under its spectral seminorm, thanks to the previous lemma. In particular, we can write A as a finite integral extension of T_d for some $d \geq 0$, and take B to be the integral closure of T_d in the normal closure of $\operatorname{Frac} A$ over $\operatorname{Frac} T_d$. This lets us break the problem into two steps.

(a) Show that if A is finite over T_d and $\operatorname{Frac} A$ is purely inseparable over $\operatorname{Frac} T_d$, then the spectral norm on A is complete.

(b) Show that if B is finite over A, $\operatorname{Frac} B$ is Galois over $\operatorname{Frac} A$, and the spectral norm on A is fully multiplicative and complete, then the spectral norm on B is complete.

So we work on these two steps separately.

(a) There is nothing to show unless the characteristic of K is $p > 0$. Let K' be the completed algebraic closure of K; then for some integer m,

$$A \subseteq K'\langle x_1^{1/p^m}, \dots, x_d^{1/p^m}\rangle$$

so it's enough to check the completeness of $K'\langle x_1^{1/p^m}, \dots, x_d^{1/p^m}\rangle$ under its spectral norm. But again, this is true because that spectral norm is the Gauss norm.

(b) Put $G = \operatorname{Gal}(\operatorname{Frac} B/\operatorname{Frac} A)$, and put

$$\operatorname{Trace}(f) = \sum_{g\in G} f^g;$$

this gives a map from $\operatorname{Frac} B$ to $\operatorname{Frac} A$ such that $\|\operatorname{Trace}(f)\|_{A,\mathrm{spec}} \leq \|f\|_{B,\mathrm{spec}}$. (This trace is the same as the trace of multiplication by f as a $\operatorname{Frac} A$-linear transformation on $\operatorname{Frac} B$.) From basic algebra, we know that the $\operatorname{Frac} A$-linear pairing

$$(x, y) \mapsto \operatorname{Trace}(xy)$$

on $\operatorname{Frac} B$ is nondegenerate.

Choose $e_1, \dots, e_n \in B$ which form a basis for $\operatorname{Frac} B$ over $\operatorname{Frac} A$, and let $e_1^*, \dots, e_n^*$ be the dual basis for the trace pairing. We show that $Ae_1 + \cdots + Ae_n$ is a Banach module under the spectral norm, by showing that the spectral norm is equivalent to the maximum norm

$$\|f_1 e_1 + \cdots + f_1 e_n\|_{B,\mathrm{spec}} = \max_i \{\|f_i\|_{A,\mathrm{spec}}\}.$$

Choose $a_0 \in A$ such that $a_0 e_j^* \in \mathfrak{o}_B^{\mathrm{spec}}$ for $j = 1, \dots, n$; now given $f_1 e_1 + \cdots + f_n e_n \in Ae_1 + \cdots + Ae_n$, we have

$$\begin{aligned} a_0 f_j &= \operatorname{Trace}(a_0 e_j^* \sum_i f_i e_i) \\ \|\operatorname{Trace}(a_0 e_j^* \sum_i f_i e_i)\|_{A,\mathrm{spec}} &\leq \|a_0 e_j^* \sum_i f_i e_i\|_{B,\mathrm{spec}} \\ &\leq \|\sum_i f_i e_i\|_{B,\mathrm{spec}}. \end{aligned}$$

Therefore

$$\|\sum_i f_i e_i\|_{B,\mathrm{spec}} \geq \|a_0\|_{A,\mathrm{spec}} \max_i \{\|f_i\|_{A,\mathrm{spec}}\}.$$

Since we also have

$$\|\sum_i f_i e_i\|_{B,\mathrm{spec}} \leq \max_i \{\|f_i\|_{A,\mathrm{spec}}\} \max_i \{\|e_i\|_{B,\mathrm{spec}}\},$$

the spectral norm restricted to $Ae_1 + \cdots + Ae_n$ is equivalent to the maximum norm, and so is a Banach norm.

For some $a \in A$, $aB \subseteq Ae_1 + \cdots + Ae_n$; since the spectral norm on A is multiplicative, the spectral norm on B is thus complete.

□

Note that this theorem can also be interpreted as follows: a sequence of elements of A converges to zero under some (any) Banach algebra norm if and only if it converges uniformly to zero on $\operatorname{Max} A$.

The reduction of an affinoid algebra

In case you are wondering when the spectral seminorm is not just a norm but is actually fully multiplicative (like the Gauss norm), here is your answer. Recall that for A an affinoid algebra, we defined

$$\mathfrak{o}_A^{\mathrm{spec}} = \{f \in A : \|f\|_{\mathrm{spec}} \leq 1\}.$$

Now define

$$\mathfrak{m}_A^{\mathrm{spec}} = \{f \in A : \|f\|_{\mathrm{spec}} < 1\}$$

and $\overline{A}^{\mathrm{spec}} = \mathfrak{o}_A^{\mathrm{spec}}/\mathfrak{m}_A^{\mathrm{spec}}$; we call the latter the *reduction* of A. Then we have the following [BGR, Proposition 6.2.3/5].

Proposition 5. *The spectral seminorm is a fully multiplicative norm if and only if A is reduced and $\overline{A}$ is an integral domain.*

Note that A being an integral domain is not enough; see exercises.

Proof. We already know that the spectral seminorm is a norm if and only if A is reduced; also, if A is reduced and the spectral norm is fully multiplicative, then the product of elements of spectral norm 1 again has spectral norm 1, so $\overline{A}$ is an integral domain. Conversely, suppose A is reduced and $\overline{A}$ is an integral domain. Given $f, g \in A$ nonzero, there exists an integer n such that $\|f\|_{\mathrm{spec}}^n$ and $\|g\|_{\mathrm{spec}}^n$ belong to $|K^*|$, by the maximum modulus principle. (Namely, the spectral seminorm is always the norm of the evaluation of f at some point whose residue field is finite over K.) Choose $c, d \in K^*$ with $c\|f\|_{\mathrm{spec}}^n = d\|g\|_{\mathrm{spec}}^n = 1$. Then the product of the images of cf^n and dg^n in $\overline{A}$ must be nonzero because $\overline{A}$ is an integral domain; that is, $\|cf^n dg^n\|_{\mathrm{spec}} = 1$. Hence

$$1 = \|cf^n dg^n\|_{\mathrm{spec}} = c\|f\|_{\mathrm{spec}}^n \cdot d\|g\|_{\mathrm{spec}}^n$$

and (by power-multiplicativity of the spectral seminorm) it follows that $\|fg\|_{\mathrm{spec}} = \|f\|_{\mathrm{spec}} \cdot \|g\|_{\mathrm{spec}}$. $\square$

Exercises

1. Give an explicit example of an affinoid algebra A which is an integral domain, but whose spectral seminorm is not fully multiplicative. (Hint: consider power series in x and x^{-1} which converge on a suitable annulus; your motivation should functions on a complex annulus which have their suprema on opposite boundary components. We'll look more at this geometric situation in the next few lectures.)