18.727, Topics in Algebraic Geometry (rigid analytic geometry)
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Even more on affinoid algebras

Ruochuan caught me last time using the following fact without justification, in the proof that spectral norms are Banach norms. In fact, this is [FvdP, Theorem 3.5.1], so I might as well explain this. (Actually, I only used part (b) in that proof, so the annoying part (a) below wasn't really needed. But let's explain it anyway.)

Theorem 1. *Let A be a reduced integral affinoid algebra. Then the integral closure of A in its fraction field is finitely generated as an A-module. (Corollary: the integral closure of A in any finite extension of its fraction field is also finite over A.)*

Proof. We can write A as a finite integral extension of some T_d by Noether normalization, so it is enough to show that the integral closure of T_d in any finite extension of its fraction field is finite over T_d. As in the previous proof, this breaks down into two steps.

(a) If L is a finite purely inseparable extension of T_d, then the integral closure of T_d in L is finite over T_d.

(b) If A is an affinoid algebra which is a normal (integrally closed in its fraction field) domain, and L is a finite Galois extension of $\operatorname{Frac} A$, then the integral closure of A in L is finite over A.

Part (b) is easy: let B be the integral closure of A in L, choose $e_1, \dots, e_n \in B$ which form a basis of L over $\operatorname{Frac} A$, and let $e_1^*, \dots, e_n^*$ be the dual basis for the trace pairing. Now simply pick $a \in A$ such that $ae_1^*, \dots, ae_n^* \in B$; then note that $af \in Ae_1 + \cdots + Ae_n$ for any $f \in B$. So B is contained in a finitely generated A-module; since A is noetherian, A is finitely generated.

Part (a) is a bit more annoying. It's clear if K is perfect, as in that case we can pass from L to the fraction field of $K\langle x_1^{1/p^n}, \dots, x_d^{1/p^n}\rangle$ for some n, and the integral closure in that field is clearly the bigger Tate algebra, which is visibly finite over T_d. You can still argue like this if $[K : K^p] < \infty$, but otherwise it gets messy. Let K' be the completed algebraic closure of K; the point is that whatever generators you get of the integral closure of $K'\langle x_1, \dots, x_d\rangle$ in $K'\langle x_1^{1/p^n}, \dots, x_d^{1/p^n}\rangle$ can be approximated by generators which are actually integral over T_d. (Compare the argument in the proof of the "closed submodule principle", i.e., [FdvP, Lemma 1.2.3], or just look this up in [FvdP, Theorem 3.5.1].)

$\square$