18.727, Topics in Algebraic Geometry (rigid analytic geometry)
Kiran S. Kedlaya, fall 2004
Tate's acyclicity theorem (amended 25 Oct 04)

The goal of this lecture is to prove Tate's acyclicity theorem (or more properly, the Gerritzen-Grauert-Tate theorem): coherent sheaves on an affinoid space are acyclic for Čech cohomology.

References: [FvdP, Section 4.2] and [BGR, 8.2]; also, [BGR, 7.3.5] for the Gerritzen-Grauert theorem. Also see Tate's original notes (Rigid analytic spaces, *Invent. Math.* **12** (1971), 257–289); more on these below. I haven't read Gerritzen and Grauert's original paper (Die Azyklizität der affinoiden Überdeckungen, in *Global Analysis, Papers in honor of K. Kodaira*, Princeton University Press, 1969, 159–184), if for no other reason than that I don't read German, so I can't comment on it.

Comments I should have made last time

All of the G-topologies on an affinoid space from last time depend only on the reduced quotient of the affinoid algebra. Namely, it's clear that the notion of a rational subspace is insensitive to nilpotents, since it's defined by evaluating functions on $\operatorname{Max} A$. But also the notion of an affinoid subspace is insensitive: if $\phi : A \to B$ is the homomorphism corresponding to an affinoid subspace of $\operatorname{Max} A$, we just take $\phi^{\mathrm{red}} : A^{\mathrm{red}} \to B^{\mathrm{red}}$ to be the homomorphism corresponding to the same subspace of $\operatorname{Max} A^{\mathrm{red}} = \operatorname{Max} A$. That means there's no harm in working only with reduced affinoid algebras for the moment (though it won't make that much difference either way).

Also, if $\phi : A \to B$ is a homomorphism of affinoids, we say ϕ is a *closed immersion* if it is surjective. Also, we say ϕ is a *locally closed immersion* (resp. *open immersion*) if the homomorphism on stalks $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ is always surjective (resp. bijective). Any map defining an affinoid subspace is a locally closed immersion (see previous handout). See [BGR, 7.3.3] for more on immersions, including some of the results that go into the argument in the next section.

A bit about the reduction process

The general idea in this handout is to reduce the checking of acyclicity for one type of coverings to checking for a simpler type. Here is the basic statement you use.

Lemma 1. *Let X be a space equipped with a G-topology, let $\mathcal{F}$ be a presheaf on X, let $\{U_i\}_{i \in I}$ be an admissible covering of X, and let $\{V_j\}_{j \in J}$ be an admissible covering of X refining $\{U_i\}$. Suppose that $\mathcal{F}$ is an acyclic sheaf on the restriction of the covering $\{V_j\}$ to each intersection $U_{i_0} \cap \cdots \cap U_{i_n}$. Then $\mathcal{F}$ is an acyclic sheaf for the covering $\{U_i\}$ if and only if it is an acyclic sheaf for the covering $\{V_j\}$.*

Proof. This is really saying that a certain Leray spectral sequence degenerates; to see it more explicitly, see [BGR, Corollary 8.1.4/3]. □

Affinoid subspaces and rational subspaces

The goal of this section is to sketch a proof of the following theorem. A complete proof appears in [BGR, 7.3.5].

Theorem 2 (Gerritzen-Grauert). *Let A be an affinoid algebra, and let X be an affinoid subspace of* $\operatorname{Max} A$*. Then A is a finite union of rational subspaces of* $\operatorname{Max} A$*.*

Beware that the converse is not true: not every finite union of rational subspaces of $\operatorname{Max} A$ is an affinoid subspace (as mentioned last time).

Let's start the argument to see where a naïve approach gets stuck. As noted above, I may assume A is reduced. Let $\phi : A \to B$ denote the representing homomorphism for X, and choose a surjection $\psi : A\langle x_1, \dots, x_n\rangle \to B$ that sections ϕ. We want to argue by induction on n, but this is a bit tricky because the image of $A\langle x_1, \dots, x_i\rangle$ in B need not be the coordinate ring of an affinoid subspace. One already sees this in the rational case: if f_0, f_1, f_2 generate the unit ideal but f_0, f_1 do not, then $A\langle x_1, x_2\rangle/(f_1 - f_0x_1, f_2 - f_0x_2)$ is the coordinate ring of an affinoid subspace but $A\langle x_1\rangle/(f_1 - f_0x_1)$ is not.

To get around this, we use a relative form of of Weierstrass preparation. Recall that a series f in $T_n = K\langle x_1, \dots, x_n\rangle$ was said to be *distinguished (in x_n) of degree d* if, when we write $f = \sum c_i x_n^i$ with $c_i \in T_{n-1}$, we have:

- c_d is a unit in T_{n-1};
- $\|c_d\|_{T_{n-1}} = \max_i\{\|c_i\|_{T_{n-1}}\}$;
- $\|c_d\|_{T_{n-1}} > \|c_i\|_{T_{n-1}}$ for $i > d$.

(Remember we said f was *normalized distinguished* if in fact $\|c_d\| = 1$, and we habitually dropped the word "normalized". However, it'll be more convenient here not to normalize.)

For $f \in A\langle x_1, \dots, x_n\rangle$ and $x \in \operatorname{Max} A$, we say f is *distinguished of degree d at x* if the image of f in $(A/\mathfrak{m}_x)\langle x_1, \dots, x_n\rangle$ is distinguished of degree d (resp. $\leq d$). We say f is *distinguished of degree d* if f is distinguished of degree d at each $x \in \operatorname{Max} A$; if instead f is distinguished of some degree $\leq d$ at each $x \in \operatorname{Max} A$, but not necessarily always the *same* degree, we say f is *distinguished of degree $\leq d$*. Note that f is distinguished of degree 0 (or ≤ 0) if and only if it is a unit.

Lemma 3. *Suppose $f \in A\langle x_1, \dots, x_n\rangle$ is distinguished of degree $\leq d$. Then the set*

$$U = \{x \in \operatorname{Max} A : f \text{ is distinguished of degree } d \text{ at } x\}$$

is a rational subspace of A.

Proof. Write $f = \sum_i c_i x_n^i$ with $c_i \in A\langle x_1, \dots, x_{n-1}\rangle$, and let f_i denote the constant coefficient of c_i. Then it is easy to check (or see [BGR, Lemma 7.3.5/7]) that

$$U = \{x \in \operatorname{Max} A : |f_i(x)| \leq |f_d(x)| \qquad (i = 0, \dots, d-1)\}.$$

Since f is distinguished of some degree at each $x \in \operatorname{Max} A$, the functions $f_0, \dots, f_d$ have no common zero, and hence generate the unit ideal. Thus U is a rational subspace, as desired. $\square$

Lemma 4 ("Relative distinction"). *Suppose the coefficients of $f \in A\langle x_1, \dots, x_n\rangle$ have no common zero on* $\operatorname{Max} A$. *Then there is an A-algebra automorphism τ of $A\langle x_1, \dots, x_n\rangle$ such that f^τ is distinguished of degree $\leq d$, for some nonnegative integer d.*

Proof. Exercise; it's pretty similar to the proof we gave when $A = K$. $\square$

Lemma 5. *Suppose that $f \in A\langle x_1, \dots, x_n\rangle$ is distinguished of degree d. Then the map*

$$A\langle x_1, \dots, x_{n-1}\rangle \to A\langle x_1, \dots, x_n\rangle/(f)$$

is finite.

Proof. As in the nonrelative case, the point is that $1, x_n, \dots, x_{n-1}^{d-1}$ generate the quotient over $A\langle x_1, \dots, x_{n-1}\rangle$. (Note that if you write f as a series in x_n, its coefficient of x_n^d has no zeroes in $\operatorname{Max} A\langle x_1, \dots, x_{n-1}\rangle$ and so is a unit.) $\square$

Now back to the proof. We have our map $\phi : A \to B$ defining an affinoid subdomain, and we chose a surjection $\psi : A\langle x_1, \dots, x_n\rangle \to B$ that sections ϕ. By Lemma 4, we can arrange for the kernel of ψ to contain a series f which is distinguished of degree $\leq d$ for some d.

Let U be the subset of $\operatorname{Max} A$ at which f is distinguished of degree d; then U is a rational subspace by Lemma 3. (It could of course be empty!) We now show that $X \cap U$ is a finite union of rational subspaces of U, and hence of $\operatorname{Max} A$. Let A' be the coordinate ring of U; then the map $A'\langle x_1, \dots, x_{n-1}\rangle \to A'\langle x_1, \dots, x_n\rangle/(f)$ is finite by Lemma 5, as then is the map

$$A'\langle x_1, \dots, x_{n-1}\rangle \to B\widehat{\otimes}_A A'.$$

However, $\psi : A' \to B\widehat{\otimes}_A A'$ represents an affinoid subspace of $U = \operatorname{Max} A'$, which means that the local rings at its points are isomorphic to the corresponding local rings at the points of $\operatorname{Max} A'$. It follows that the map $A'\langle x_1, \dots, x_{n-1}\rangle \to B\widehat{\otimes}_A A'$, being finite but not inducing any nontrivial extensions of local rings, must actually be surjective. (For more clarification, see [BGR, Proposition 7.3.3/8].) So we may apply the induction hypothesis to conclude that $X \cap U$ is a finite union of rational subspaces $U_1 \cup \cdots \cup U_s$.

To finish, we need one more lemma.

Lemma 6. *Let X be an affinoid subspace of* $\operatorname{Max} A$, *and let U be the rational subspace given by*

$$U = \{x \in \operatorname{Max} A : |f_i(x)| \leq |f_0(x)| \qquad (i = 1, \dots, m)\},$$

for some $f_0, \dots, f_m$ generating the unit ideal in A. Suppose that $U \subseteq X$. Then for some $\epsilon > 1$ in the divisible closure of $|K^*|$, the rational subspace

$$U_\epsilon = \{x \in \operatorname{Max} A : |f_i(x)| \leq \epsilon |f_0(x)| \qquad (i = 1, \dots, m)\}$$

has the property that $X \cap U_\epsilon$ is also a rational subspace.

Proof. The proof is an approximation argument adapted from [BGR, Extension Lemma 7.3.4/10]. Again, let $\psi : A\langle x_1, \dots, x_n\rangle \to B$ be a surjection sectioning the map $A \to B$. Since $U \subseteq X$, we can choose $a_1, \dots, a_n \in A$ such that $|a_i - f_0^N x_i|_U < 1$ for some $N \in \mathbb{N}$. By a continuity argument (see exercises, or [BGR, Proposition 7.3.4/8]), we can choose $\epsilon > 1$ such that $|a_i - f_0^N x_i|_{U_\epsilon} < 1$ for all i, and such that f_0 has no zeroes on U_ϵ.

If we then write A_ϵ and B_ϵ for the coordinate rings of U_ϵ and $X \cap U_\epsilon$, respectively, then the map $A_\epsilon \to B_\epsilon$ factors as

$$A_\epsilon \to A_\epsilon\langle y_1, \dots, y_n\rangle/(y_1 - f_0^N x_1, \dots, a_n - f_0^N x_n) \to B_\epsilon.$$

and that the latter map is surjective. Thus we can cut out $X \cap U_\epsilon$ within U_ϵ by imposing the conditions that $|a_i| \leq 1$ and that some other functions actually vanish. □

Using the previous lemma, we can grow each of $U_1, \cdots, U_s$ slightly to some $U_1', \dots, U_s'$ and still know that each $X \cap U_i'$ is a union of rational subspaces. By so doing, we can ensure that the complement $X \setminus (U_1' \cup \cdots \cup U_n')$ is contained in a union of rational subspaces V of $\operatorname{Max} A$ which is in turn contained in the subspace of $\operatorname{Max} A$ on which f is *not* distinguished of degree d, i.e., is distinguished of degree $\leq d-1$. (That is, each U_i' is a "strict neighborhood" of U_i, and this ensures that $X \setminus (U_1' \cup \cdots \cup U_n')$ is contained in a union of rational subspaces of $\operatorname{Max} A$ not meeting U. See next mini-section for more details.) By induction on d, we can cover $X \cap V$ with rational subspaces, completing the argument.

Strict neighborhoods

Here's a clarification of the last argument in the previous section. For U an admissible open in a strong G-topological space X having the property that $X \setminus U$ is also admissible, we say an admissible open V containing U is a *strict neighborhood* of U in X if the covering $\{V, X \setminus U\}$ of X is admissible.

We then have the following lemma; compare this to the examples of admissible covers we discussed for $\mathbb{P}$.

Lemma 7. *Let A be an affinoid algebra, and let U be the rational subspace of* $\operatorname{Max} A$ *given by*

$$U = \{x \in \operatorname{Max} A : |f_i(x)| \leq |f_0(x)| \qquad (i = 1, \dots, m)\},$$

for some $f_0, \dots, f_m$ generating the unit ideal in A. For $\epsilon > 1$ in the divisible closure of $|K^*|$, define the rational subspace

$$U_\epsilon = \{x \in \operatorname{Max} A : |f_i(x)| \leq \epsilon|f_0(x)| \qquad (i = 1, \dots, m)\}.$$

Then an admissible open subset V of $\operatorname{Max} A$ *for the strong G-topology is a strict neighborhood of U if and only if it contains some U_ϵ. Moreover, if V is a strict neighborhood, then there is a finite union of rational subspaces of* $\operatorname{Max} A$*, contained in* $\operatorname{Max} A \setminus U$*, which together with some affinoid subspace contained in U form a cover (automatically admissible) of* $\operatorname{Max} A$*.*

Proof. We first check that each U_ϵ is a strict neighborhood of U. Choose δ in the divisible closure of $|K^*|$ with $\epsilon > \delta > 1$; for $j = 1, \dots, m$, put

$$V_{\delta,j} = \{x \in \operatorname{Max} A : \delta|f_0(x)| \leq |f_j(x)|, |f_i(x)| \leq |f_j(x)| \qquad (i \neq j)\}.$$

Then $V_{\delta,j}$ is rational: if $\delta^N = |c|$ for $c \in K$, then $V_{\delta,j}$ is the rational subspace defined by the functions f_j^N, cf_0^N and the f_i^N for $i \neq j$. The union of the $V_{\delta,j}$ consists of those x for which $\max_{i>0}\{|f_i(x)|\} \geq \delta|f_0(x)|$, so this plus U_ϵ covers $\operatorname{Max} A$. This gives an admissible cover refining the cover $\{U_\epsilon, \operatorname{Max} A \setminus U\}$, so U_ϵ is a strict neighborhood.

Conversely, suppose V is a strict neighborhood of U_ϵ. Then there exist finitely many affinoid subspaces of $\operatorname{Max} A$ contained in the complement of U, whose union together with V covers $\operatorname{Max} A$. Let $W_1, \dots, W_r$ be those subspaces; then the function

$$\frac{|f_0(x)|}{\max_{i>0}\{|f_i(x)|\}}$$

takes a maximum value on each W_j, by a suitable application of the maximum modulus principle. If the maximum over all of $W_1, \dots, W_r$ is $\delta < 1$, then $U_\epsilon \subseteq V$ whenever $\epsilon < \delta$. □

Acyclicity of affinoid coverings

By the Gerritzen-Grauert theorem, we know that every finite covering of an affinoid space by affinoid subspaces can be refined to a finite covering by rational subspaces. Tate proved that finite rational coverings are acyclic for Čech cohomology, by which it follows that they are also acyclic for finite affinoid coverings. The latter is known as "Tate's acyclicity theorem", though the name "Gerritzen-Grauert-Tate theorem" would be more apt.

First, however, we need some further simplification of the types of coverings we are using. Let A be an affinoid algebra. Given $f_1, \dots, f_n \in A$ having no common zero (i.e., generating the unit ideal), put

$$U_i = \{x \in \operatorname{Max} A : |f_j(x)| \leq |f_i(x)| \qquad (j \neq i)\}.$$

Then $U_1, \dots, U_n$ form a covering of $\operatorname{Max} A$ by rational subspaces; such a covering is called a *standard rational covering* of $\operatorname{Max} A$.

Lemma 8. *Every finite covering of* $\operatorname{Max} A$ *by affinoid subspaces can be refined to a standard rational covering.*

Proof. By the Gerritzen-Grauert theorem, we may assume without loss of generality that we are starting with a finite covering by rational subspaces $V_1, \dots, V_m$. Suppose V_i is defined by the inequalities $|g_{ij}(x)| \leq |g_{i0}(x)|$ for $g_{ij} \in A$ generating the unit ideal, with j running from 1 to some n_i. Now take the f's to be the products of the form

$$g_{1j_1} \cdots g_{nj_n} \qquad (1 \leq j_i \leq n_i)$$

in which some j_i is equal to 0; these have no common zero because g_{i0} has no zero on V_i, and the V_i cover X. Moreover, the standard cover given by these f's refines the given cover (exercise). □

In fact, we can do better. Given $f_1, \dots, f_n \in A$, put

$$U_i^{\leq} = \{x \in \operatorname{Max} A : |f_i(x)| \leq 1\}, \qquad U_i^{\geq} = \{x \in \operatorname{Max} A : |f_i(x)| \geq 1\}.$$

Then the collection of sets of the form $U_1^* \cap \cdots \cap U_n^*$, where each $* \in \{\leq, \geq\}$, form a covering of $\operatorname{Max} A$; such a covering is called a *Laurent covering* of $\operatorname{Max} A$. (You might be surprised that such a covering is permitted! After all, in $\mathbb{P}$ the analogous thing would have been the closed unit disc and its inverse, which only meet along their boundary. But in fact that is indeed an admissible cover, since both spaces are rational! The point is that glueing two discs along their "boundary" is completely reasonable from our point of view, since that boundary is huge–and rational.)

Lemma 9. *Every finite covering of* $\operatorname{Max} A$ *by affinoid subspaces can be refined to a Laurent covering.*

Proof. By Lemma 8, it suffices to start with a standard rational covering, say the one generated by $f_1, \dots, f_n$. By the maximum modulus principle, there exists $c \in K^*$ such that

$$|c|^{-1} < \inf_{x \in X} \{\max_i \{|f_i(x)|\}\}.$$

(More explicitly, the subspace on which $|f_i(x)|$ is maximized by $i = j$ is a rational subspace, hence affinoid, and $|f_i(x)|$ achieves its maximum there by the maximum modulus principle.) The point then is that

$$\max_i \{|cf_i(x)|\} > 1 \qquad \text{for all } x \in \operatorname{Max} A.$$

Then the Laurent covering generated by $cf_1, \dots, cf_n$ may not refine the original standard covering, but it has the following convenient property: for each U in my new covering, the restriction of my old (standard rational) covering to U is a standard rational covering generated by *units* on U. (If U is the open consisting of those x where $|f_i(x)| \geq 1$ for $i \in S$ and $|f_i(x)| \leq 1$ for $i \notin S$, then the restriction of the old covering to U is the standard rational covering generated by the cf_i for $i \in S$, which are units in U.)

It thus suffices to check that a standard rational covering generated by units $g_1, \dots, g_m$ can be refined to a Laurent covering. But this is easy: just use the functions g_i/g_j for $1 \leq i < j \leq m$ as the generators (exercise). □

Theorem 10 (Acyclicity theorem for the structure sheaf). *Let* $X = \operatorname{Max} A$ *be an affinoid space. Then for any finite covering of* X *by affinoid subspaces and any finitely generated* A*-module* M*, the presheaf* $\mathcal{M}$ *on* X *(for the somewhat weak* G*-topology) associated to* M*, whose sections on an affinoid space* U *with coordinate ring* B *are* $M \otimes_A B$*, is a sheaf and its higher Čech cohomology spaces vanish.*

Proof. By Leray's theorem and Lemma 8, it is enough to check this for Laurent coverings. In fact, it is enough to check a Laurent covering generated by a single element. (If you want to see this reduction written out in more detail, see [BGR, Chapter 8].)

I'll first check for $M = A$, i.e., $\mathcal{M} = \mathcal{O}$. Say the Laurent covering is generated by $f \in A$. By the same argument as we used for $\mathbb{P}$, it suffices to show that

$$0 \to A \to A\langle f\rangle \oplus A\langle f^{-1}\rangle \xrightarrow{d^0} A\langle f, f^{-1}\rangle \to 0 \tag{1}$$

is exact, where

$$\begin{aligned} A\langle f\rangle &= A\langle x\rangle/(x-f) \\ A\langle f^{-1}\rangle &= A\langle y\rangle/(yf-1) \\ A\langle f, f^{-1}\rangle &= A\langle x, y\rangle/(x-f, yf-1) \end{aligned}$$

and d_0 is the difference between the two canonical maps. The exactness of this sequence can be verified by chasing through the diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & (x-f)A\langle x\rangle \oplus (yf-1)A\langle y\rangle \arrow[r] \arrow[d] & (x-f)A\langle x,y\rangle/(xy-1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & A\langle x\rangle \oplus A\langle y\rangle \arrow[r] \arrow[d] & A\langle x,y\rangle/(xy-1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & A\langle f\rangle \oplus A\langle f^{-1}\rangle \arrow[r, "d_0"] \arrow[d] & A\langle f, f^{-1}\rangle \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

by checking exactness of the first and second rows, and of all of the columns.

In fact, the sequence (1) is not only exact, but split: there is a splitting induced by continuity from the map $A[x, y] \to A\langle f\rangle \oplus A\langle f^{-1}\rangle$ that sends $x^i y^j$ to x^{i-j} if $i \geq j$ and to y^{j-i} if $i < j$. (That is, extend by continuity to a map on $A\langle x, y\rangle$ and note that the ideal $(x - f, yf - 1)$ is contained in the kernel.) That means the sequence remains exact upon tensoring over A by any A-module M, so we get the desired result for any M. □

Pay careful attention to where we used the fact that M is finitely generated: it's because we only tensored in the last step. If M were not finitely generated, we would also have to have completed the tensor product, and as has been pointed out before, completing tensor products over arbitrary Banach algebras is a somewhat unpredictable operation.

Any sheaf arising from a finite A-module is called a *coherent sheaf*; note that it also gives a sheaf on the strong topology by abstract G-topology properties. (Less abstract, you compute the sections on an arbitrary open by covering it admissibly with affinoid subspaces, or even rational subspaces, and glueing sections on those together.) Tate's theorem implies on one hand that coherent sheaves can be specified on a finite cover by affinoid subspaces (by providing modules and glueing isomorphisms), and on the other hand that they always have trivial higher Čech cohomology. Life is good.

Next time: we are now ready to start glueing affinoids together to form rigid analytic spaces!

Historical note: Tate's notes

Warning: this "history" is mostly secondhand (or thirdhand), so don't rely on this too heavily.

Tate's original notes, in which he proves the acyclicity theorem for coverings of an affinoid space by rational subsets (or rather, by "special affine subsets" in his terminology, but the result is equivalent), are conventionally dated to 1962, when he lectured on the subject of rigid analytic spaces at Harvard. The subject took off like a shot after that, but Tate only distributed his notes privately and steadfastly refused to publish them. At some point, however, Tate's trusted chain of custody broke down, and the notes came into the possession of the editors of *Inventiones*, who ultimately decided (to posterity's benefit) that these should appear in print. (I believe what happened is that someone stole the notes out of the drawer in his office where he kept them, but I don't have a corroboration for this handy. And even if I did, I wouldn't tell who did it!)

While Tate did introduce the concept of an affinoid subspace (or "affine subset" in his terminology), he did not even formulate the question of whether an arbitrary finite covering by affinoid subspaces is acyclic. This was resolved by Gerritzen and Grauernt, who form part of the "German school" that developed rigid analytic geometry more fully in the 1960s. (Besides the aforementioned, and also the authors of [BGR], this school most notably includes Kiehl, who proved some key finiteness theorems which we will touch on a bit later.)

Exercises

1. Prove the "relative distinction lemma" (Lemma 4). (Hint: see [BGR, Proposition 7.3.5/9].)

2. Let A be an affinoid algebra, let $f_0, \dots, f_n \in A$ generate the unit ideal, and for ϵ in the divisible closure of $|K^*|$, put
$$U_\epsilon == \{x \in \operatorname{Max} A : |f_i(x)| \leq \epsilon |f_0(x)| \qquad (i = 1, \dots, m)\}.$$
Prove that for any $g \in A$, the function $\epsilon \mapsto |g|_{U_\epsilon}$ extends to a continuous function from $\mathbb{R}_{>0}$ to $\mathbb{R}_{\geq 0}$.

3. Complete the verifications of Lemma 8 and Lemma 9.

4. Prove that any affinoid subset of $\mathbb{P}$ is rational. (Hint: first check that our old concept of "rational subspace of $\mathbb{P}$", restricted to subsets of the closed unit disc, is consistent with our new concept of a rational subspace. Then apply Gerritzen-Grauert to reduce to checking that a finite union of rational subspaces of the closed unit disc is rational.)