**18.727, Topics in Algebraic Geometry (rigid analytic geometry)**
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**Tate algebras (or, Commutative algebra revisited)**

We will now talk a bit about Tate algebras, which play a role like that of the polynomial rings over a field in ordinary algebraic geometry.

As usual, $K$ is a complete ultrametric field, which you may assume is discretely valued if you prefer, and $k$ is its residue field. Reminder: I write $\mathfrak{o}_K$ and $\mathfrak{m}_K$ for the valuation subring of $K$ and its maximal ideal, rather than the bizarre $K^o$ and $K^{oo}$ used in [FvdP].

Convention: when I write $\sum_I c_I x^I$, the sum will be running over tuples (normally of nonnegative integers) $I = (i_1, \dots, i_n)$, and $x^I = x_1^{i_1} \cdots x_n^{i_n}$.

**References:** The main reference is [FvdP, Section 3.1], but I have lots of issues with the presentation, so I've supplemented from [BGR, 5.1]. Also, I'll cite Lang's *Algebra* (third edition; numbering may differ in the current edition) in the exercises as [L].

## Tate algebras

The *Tate algebra* (or *standard affinoid algebra*)

$$T_n = T_{n,K} = K\langle x_1, \dots, x_n \rangle$$

is the subring of the ring of formal power series $K[\![x_1, \dots, x_n]\!]$ consisting of sums $\sum_I c_I x^I$ such that $|c_I| \to 0$ as $I \to \infty$; what that really means is that for any $\epsilon > 0$, there are only finitely many tuples $I$ such that $|c_I| > \epsilon$. Such series are sometimes called *strictly convergent power series* (as in [BGR]).

Define the *Gauss norm* on $T_n$ by the formula

$$\left\| \sum_I c_I x^I \right\| = \max_I \{|c_I|\};$$

note that the max is really a max and not a supremum, since the $|c_I|$ tend to 0. This is in fact a norm, under which $T_n$ becomes a Banach algebra over $K$ of countable type (since the monomials $x^I$ have dense span); in fact, it's isomorphic as a Banach space to the space $c_0$ of null sequences. (Trivial but handy consequence of this definition: the image of the Gauss norm is the same as the image of $K$ under its norm. This will let us do some "normalization" arguments.)

As usual for normed rings, I write $\mathfrak{o}_{T_n}$ to mean the subring of $T_n$ consisting of elements of norm $\leq 1$, and $\mathfrak{m}_{T_n}$ to mean the ideal of $\mathfrak{o}_{T_n}$ consisting of elements of norm $< 1$. Then $\mathfrak{o}_{T_n}/\mathfrak{m}_{T_n} = k[x_1, \dots, x_n]$; given $f \in \mathfrak{o}_{T_n}$, I'll write $\overline{f}$ for its image in $k[x_1, \dots, x_n]$ and call it the *reduction* of $f$. (That's why I don't use the overbar for algebraic closures!)

Here are a few basic facts about Tate algebras.

**Lemma 1.** *(a) A series $\sum c_I x^I \in K[\![x_1, \dots, x_n]\!]$ belongs to $T_n$ if and only if the sum $\sum c_I \alpha_1^{i_1} \cdots \alpha_n^{i_n}$ converges in $K$ for any $\alpha_1, \dots, \alpha_n \in \mathfrak{o}_K$.*

(b) *Suppose that the residue field $k = \mathfrak{o}_K/\mathfrak{m}_K$ is infinite. Then given a series $\sum c_I x^I \in T_n$, there exist $\alpha_1, \dots, \alpha_n \in \mathfrak{o}_K$ such that*

$$\left\| \sum_I c_I x^I \right\| = \left| \sum_I c_I \alpha_1^{i_1} \cdots \alpha_n^{i_n} \right|.$$

(c) *The Gauss norm is* fully *multiplicative: for all $f, g \in T_n$, $\|fg\| = \|f\| \cdot \|g\|$. (Remember, a Banach algebra is only required to have $\|fg\| \leq \|f\| \cdot \|g\|$.)*

*Proof.* (a) This is clear; you need only check $\alpha_1 = \cdots = \alpha_n = 1$.

(b) We may as well assume $\sum_I c_I x^I \neq 0$. Let $P(x_1, \dots, x_n)$ be the sum of $c_I x^I$ over all tuples $I$ for which $|c_I|$ is maximal. Then $P$ is a polynomial in $x_1, \dots, x_n$, $\|P\| = \|\sum_I c_I x^I\|$, and

$$\|P(x_1, \dots, x_n) - \sum_I c_I x^I\| < \|P\|.$$

It thus suffices to prove the claim for $P$ instead of the original series. But $P$ can be written as the product of some $c \in K$ with a polynomial $P_0$ which has coefficients in $\mathfrak{o}_K$ but not all in $\mathfrak{m}_K$. Since $k$ is infinite, the reduction $\overline{P_0}$ does not vanish everywhere. Pick $\alpha_1, \dots, \alpha_n \in \mathfrak{o}_K$ so that $P_0(\alpha_1, \dots, \alpha_n) \notin \mathfrak{m}_K$; then these are a good choice.

(c) Write $f = \sum_I c_I x^I$, $g = \sum_J d_J x^J$, and $fg = \sum_H e_H x^H$. Let $I$ be the tuple maximizing $|c_I|$ which is first in lexicographic order. (That is, you compare tuples by first comparing their first components, then their second if the firsts are tied, then their thirds, and so on.) Likewise, let $J$ be the tuple maximizing $|d_J|$ which is first in lexicographic order. Then $e_{I+J}$ equals $c_I d_J$ plus some other terms of the form $c_{I'} d_{J'}$, where $I'$ and $J'$ are two other tuples adding up to $I + J$. But that means that either $I'$ must precede $I$ in lexicographic order, in which case $|c_{I'}| < |c_I|$ and $|d_{J'}| \leq |d_J|$, or $J'$ must precede $J$ in lexicographic order, in which case $|c_{I'}| \leq |c_I|$ and $|d_{J'}| \leq |d_J|$. In any case, we see that $|e_{I+J} - c_I d_J| < |c_I d_J|$, so $|e_{I+J}| = |c_I d_J$. It follows that $\|fg\| \geq \|f\| \cdot \|g\|$; since we already know the other inequality, we have $\|fg\| = \|f\| \cdot \|g\|$.

□

It is worth noting what the units are in $T_n$; since we can normalize in $T_n$, we just treat units of norm 1. (This is [BGR, Proposition 5.1.3/1].)

**Lemma 2.** *For $f \in T_n$ with $\|f\| = 1$, the following are equivalent.*

(a) *$f$ is a unit in $\mathfrak{o}_{T_n}$.*

(b) *$f$ is a unit in $T_n$.*

(c) *$\overline{f}$ is constant (i.e., is a unit in $k[x_1, \dots, x_n]$).*

(d) *$|f(0)| = 1$ and $\|f - f(0)\| < 1$.*

*Proof.* Note that (a) and (b) are equivalent because $\|\cdot\|$ is fully multiplicative. Clearly (a) implies (c), and (c) and (d) are equivalent. Finally, given (d), the series $f(0)\sum_i (1 - f/f_0)^i$ converges to a reciprocal of $f$ in $\mathfrak{o}_{T_n}$, so (a) follows. $\square$

Also, note that there is an isomorphism

$$K\langle x_1, \dots, x_m\rangle \widehat{\otimes} K\langle y_1, \dots, y_n\rangle \cong K\langle x_1, \dots, x_m, y_1, \dots, y_n\rangle.$$

The Tate algebra is meant to be the "ring of functions on the closed unit polydisc". Its properties are a blend between properties of polynomial rings and formal power series rings. This hybrid nature makes it possible to do analytic geometry using Tate algebras using a lot of our intuition from algebraic geometry.

## Weierstrass preparation

We say $f \in \mathfrak{o}_{T_n}$ is *(normalized) distinguished (in $x_n$) of degree $d$* if

$$\overline{f} = c_0 + c_1 x_n + \cdots + c_d x_n^d$$

with $c_d \in k^*$ and $c_i \in \mathfrak{o}_{T_n}$ for $i = 0, \dots, d-1$. Note that $\|f\| = 1$, whence the "normalized"; if you allow something of this form times an element of $K^*$, you get elements which are *distinguished* in the terminology of [BGR]. However, for this lecture, all my distinguished elements will also be normalized, so I won't keep saying "normalized". (Terminology rant: [FvdP] use "regular" for my "normalized distinguished", but the word "regular" will come up later with a more useful meaning.)

Then one has the following results; see [FvdP, Theorem 3.1.1] and [BGR, 5.2]. (Caution: [FvdP] incorrectly applies the moniker "preparation" to (c) instead of (b).)

**Theorem 3.** *(a) (Division algorithm) Suppose $f \in \mathfrak{o}_{T_n}$ is distinguished in $x_n$ of degree $d$. Then any $g \in T_n$ can be uniquely written as $qf + r$ such that $q \in T_n$ and $r \in T_{n-1}[x_n]$, where the degree of $r$ in $x_n$ is less than $d$. Moreover, $\|g\| = \max\{\|q\|, \|r\|\}$.*

*(b) (Preparation) If $f \in \mathfrak{o}_{T_n}$ is distinguished in $x_n$ of degree $d$, then there is a unique way to write $f = gh$ with $g \in \mathfrak{o}_{T_{n-1}}[x_n]$ monic of degree $d$ (and hence distinguished) and $h \in \mathfrak{o}_{T_n}^*$.*

*(c) (Distinction) If $f_1, \dots, f_m \in \mathfrak{o}_{T_n}$ all have norm $1$, then there exists an automorphism $\tau$ of $T_n$ (preserving Gauss norms) such that $f_1^\tau, \dots, f_m^\tau$ are distinguished in $x_n$.*

The last parenthetical is actually moot, as all automorphisms of $T_n$ will preserve the Gauss norm, but this will only become obvious a bit later.

*Proof.* (a) We first verify uniqueness. If $qf + r = q'f + r'$ are two different decompositions of the same $g$, then $(q - q')f = r' - r$. By the full multiplicativity of the Gauss norm, this means $\|q - q'\| = \|r' - r\|$; pick some $c \in K$ with $|c| = \|q - q'\|^{-1}$. Then

$c(q-q')f = c(r'-r)$ and so the same is true with bars everywhere; but that contradicts uniqueness in the ordinary division algorithm for polynomials.

We next verify the claim about the norm. If $g = qf + r$, then on one hand

$$\|g\| \leq \max\{\|qf\|, \|r\|\} = \max\{\|q\|, \|r\|\},$$

and we know we have equality if $\|q\| \neq \|r\|$. But if $\|q\| = \|r\| > \|g\|$, we can choose $c \in K$ with $|c| = \|q\|^{-1}$, and then $cg = cqf + cr$. But the reduction of $cg$ is zero, so by the uniqueness in the ordinary division algorithm, $cq$ and $cr$ would also have to have zero reductions, contradiction. So even in case $\|q\| = \|r\|$ we must have $\|g\| = \max\{\|q\|, \|r\|\}$.

Now for existence. We first check this in case $f = f_0 = c_0 + c_1 x_n + \cdots + c_d x_n^d$ with each $c_i \in \mathfrak{o}_{T_{n-1}}$; this forces $c_d \in \mathfrak{o}_K^*$. Put $g = \sum d_I x^I$, and apply the ordinary ordinary division algorithm to write $x^I = q_I f + r_I$ with $q_I, r_I \in T_{n-1}[z_n]$, with the degree of $r_I$ in $z_n$ being less than $d$. By what we already showed, we have $\max\{\|q_I\|, \|r_I\|\} = \|x^I\| = 1$; thus the series

$$q = \sum_I d_I q_I, \qquad r = \sum_I d_I r_I$$

converge in $T_n$ and $T_{n-1}[z_n]$, respectively (the latter because I've bounded the degrees, so I really get a polynomial in $z_n$ and not a series). By design, $g = qf + r$.

Now for the general case; write $f = f_0 - D$ where $f_0$ is as in the previous case and $\|D\| < 1$. Given $g$, we put $g_0 = g$; given $g_i$, apply what I just did to write

$$g_i = q_i f_0 + r_i = q_i f + r_i + q_i D$$

and put $g_{i+1} = q_i D$. Then $q = \sum_i q_i$ and $r = \sum r_i$ converge to limits satisfying $g = qf + r$, and $r$ is again a polynomial in $z_n$ of degree less than $d$.

(b) We first check existence. Apply division to obtain $q', r'$ such that $x_n^d = q'f + r'$, and put $q = x_n^d - r'$; then $q \in \mathfrak{o}_{T_{n-1}}[x_n]$ is monic of degree $d$ and $q'f = q$. On reductions, we have $\overline{q} = \overline{q'f}$, and $\overline{q}$ and $\overline{f}$ are polynomials of the same degree. Hence $\overline{q'}$ is a unit, and so $q'$ is a unit by Lemma 2. We can thus factor $f = gh$ with $g = q$ and $h = (q')^{-1}$.

We next check uniqueness. If $f = gh$ is a presentation of the desired form, we have

$$x_n^d = h^{-1} f + (x_n^d - g),$$

and this is what you get from an application of (a). Thus specifying $f$ uniquely determines $x_n^d - g$, and hence determines $g$ and $h$.

(c) I'll let you find the easy proof for $k$ infinite for yourself; instead, I'll give the slightly more elaborate argument that also works for $k$ finite. Write $f_l = \sum c_{I,l} t^I$ for $l = 1, \ldots, m$, and choose integers $e_1, \ldots, e_{n-1} \geq 0$ such that whenever $I \neq J$ are among the finitely many tuples with $|c_{I,l}| = |c_{J,l}| = 1$ for some $l$, we have

$$e_1 i_1 + \cdots + e_{n-1} i_{n-1} + i_n \neq e_1 i_j + \cdots + e_{n-1} j_{n-1} + j_n.$$

(For instance, you can choose the $e_i$ to be successive powers of a bigger than any value of $i_1, \dots, i_n$ that shows up in the tuples.) Let $\tau$ be the automorphism which substitutes $x_i + x_n^{e_i}$ in place of $x_i$ for $i = 1, \dots, n-1$ (and fixes $x_n$). Then

$$\overline{f_l^\tau} = \sum \overline{c_{I,l}} (x_1 + x_n^{e_1})^{i_1} \cdots (x_{n-1} + x_n^{e_{n-1}})^{i_{n-1}} x_n^{i_n};$$

if you pick out the unique tuple $i_1, \dots, i_n$ maximizing $e = e_1 i_1 + \cdots + e_{n-1} i_{n-1} + i_n$, then in the reduction, the unique term of highest degree that you see is $x_n^e$. That means each $f_l^\tau$ is distinguished.

□

## More properties of Tate algebras

Weierstrass preparation immediately yields the analogue of the Hilbert basis theorem for Tate algebras; in the case of $K$ discretely valued, this is a theorem of Fulton. (Yes, *that* Fulton! See: A note on weakly complete algebras, *Bull. Amer. Math. Soc.* **75** (1969), 591–593.)

**Proposition 4 (Hilbert basis theorem).** *The ring $T_n$ is noetherian.*

*Proof.* Induction on $n$. Given a nonzero ideal $I$ of $T_n$, choose $f \in I$ nonzero, and apply distinction to find an automorphism $\tau$ of $T_n$ such that $f^\tau$ is distinguished in $x_n$ of some degree $d$. Using division, we see that $I^\tau$ is generated by $f^\tau$ together with $I^\tau \cap T_{n-1}[z_n]$. By the induction hypothesis, $T_{n-1}$ is noetherian, as then is $T_{n-1}[z_n]$ by the usual Hilbert basis theorem. Thus $I^\tau$ is finitely generated, as then is $I$. □

**Proposition 5.** *The ring $T_n$ is a unique factorization domain.*

*Proof.* Again, induct on $n$. Given $f \in T_n$, suppose $f = g_1 \cdots g_m = h_1 \cdots h_n$ are two factorizations of $f$ into irreducibles; by pushing scalars around, we may reduce to the case where $\|g_i\| = \|h_j\| = 1$ for all $i, j$. By distinction, there exists an automorphism $\tau$ of $T_n$ such that $f$, the $g_i^\tau$, and the $h_j^\tau$ are all distinguished. By preparation, we can write each $g_i^\tau = P_i u_i$ with $P_i \in \mathfrak{o}_{T_{n-1}}[x_n]$ monic and $u_i \in \mathfrak{o}_{T_n}^*$, and likewise write $h_j^\tau = Q_j v_j$ with $Q_j \in \mathfrak{o}_{T_{n-1}}[x_n]$ monic and $v_j \in \mathfrak{o}_{T_n}^*$. Then $P_1 \cdots P_m$ equals $Q_1 \cdots Q_n$ times a unit, but both sides are monic polynomials in $x_n$ over $\mathfrak{o}_{T_{n-1}}$, necessarily of the same degree (since that's true on the reductions). Thanks to the uniqueness of the preparation of $P_1 \cdots P_m$, we must in fact have $P_1 \cdots P_m = Q_1 \cdots Q_n$. Moreover, each $P_i$ and $Q_j$ is irreducible in $T_n$, hence also in $T_{n-1}[x_n]$; thus the factorizations agree up to units, by the unique factorization theorem for polynomials over a UFD (i.e., unique factorization over a field plus "Gauss's lemma"). That proves that the original factorizations of $f^\tau$, and hence of $f$, agree up to units. □

**Proposition 6.** *The Krull dimension of $T_n$ is $n$.*

*Proof.* The sequence of prime ideals

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \cdots (x_1, x_2, \cdots, x_n)$$

shows that the Krull dimension is at least $n$. On the other hand, for any irreducible $f \in T_n$, by distinction plus preparation, $T_n/(f)$ is finite over $T_{n-1}$, and so has Krull dimension $n-1$. (Remember that making a finite ring extension of a noetherian ring cannot increase its Krull dimension.) Hence $T_n$ has Krull dimension at most $n$, yielding the claim. $\square$

## Affinoid algebras and Noether normalization

An *affinoid algebra* is a $K$-algebra $A$ of the form $T_n/\mathfrak{a}$ for some ideal $\mathfrak{a}$. By Fulton's theorem, $A$ is noetherian.

Note that there are a couple of minor discrepancies between this definition and the one in [FvdP, 3.1]. For one, they use the term "Tate algebra" to mean *any* affinoid algebra; I have seen this elsewhere, but I still think it is nonstandard (e.g., [BGR] does not do this). For another, they define affinoid algebras as integral extensions of Tate algebras. This amounts to using the business end of the Noether normalization theorem (see below), and strikes me as bizarre.

**Proposition 7 (Noether normalization).** *Let $\mathfrak{a}$ be an ideal of $T_n$, and let $A = T_n/\mathfrak{a}$ be the corresponding affinoid algebra. Then there exists a finite injective map $T_d \to A$ for some $d$; moreover, the Krull dimension of $A$ is equal to $d$.*

*Proof.* We first prove the existence of the map, by induction on $n$. We may as well assume $\mathfrak{a}$ is a nontrivial ideal; by distinction and preparation, after applying an automorphism of $T_n$ we may assume that $\mathfrak{a}$ contains a monic polynomial $f \in T_{n-1}[x_n]$. Then $T_n/(f)$ is finite over $T_{n-1}$; if we put $\mathfrak{b} = \mathfrak{a} \cap T_{n-1}$, then $T_n/\mathfrak{a}$ is finite over $T_{n-1}/\mathfrak{b}$. By the induction hypothesis, $T_{n-1}/\mathfrak{b}$ is finite over some $T_d$, yielding the claim.

For the statement about the Krull dimension, we need only recall that for $A \to B$ a finite injective homomorphism of noetherian rings, the rings $A$ and $B$ have the same Krull dimension, and that the Krull dimension of $T_d$ is $d$ by Proposition 6. $\square$

Warning: unlike in the polynomial situation, an affinoid algebra can have an affinoid subalgebra of greater Krull dimension! See [FvdP, Exercises 3.2.2].

Note that the Nullstellensatz for Tate algebras falls out as a consequence.

**Corollary 8.** *For any maximal ideal $\mathfrak{m}$ of $T_n$, the field $T_n/\mathfrak{m}$ is finite over $K$.*

*Proof.* A field has Krull dimension 0, so by Noether normalization, there must exist a finite map $K = T_0 \to T_n/\mathfrak{m}$. $\square$

## Affinoid algebras are Banach algebras (and canonically so!)

Recall the following lemma (Lemma 1 from "functional analysis, part 2").

**Lemma 9.** *Let $A$ be a Banach algebra over $K$ which is noetherian as a ring. Let $M$ be a Banach module over $A$ which is module-finite over $A$. Then any $A$-submodule of $M$ is closed.*

In particular, since $T_n$ is noetherian, any ideal of $T_n$ is closed. Hence any affinoid algebra $A$ inherits from a presentation $T_n \to A$ a quotient norm, under which it becomes a Banach algebra.

It turns out that the topology of an affinoid algebra is uniquely determined by its $K$-algebra structure, and all $K$-algebra homomorphisms of affinoid algebras. To see this, we need to back up and do a little more functional analysis; the following is [BGR, Proposition 3.7.5/2].

**Proposition 10.** *Let $B$ be a noetherian Banach algebra over $K$, and suppose there exists a collection $S$ of ideals of $B$ such that:*

*(a) for any $\mathfrak{b} \in S$, $\dim_K B/\mathfrak{b} < \infty$;*

*(b) $\cap_{\mathfrak{b} \in S} \mathfrak{b} = (0)$.*

*Then any $K$-algebra homomorphism of a noetherian Banach algebra $A$ over $K$ into $B$ is continuous (hence bounded).*

*Proof.* Let $f : A \to B$ be a $K$-algebra homomorphism; we will apply the closed graph theorem to the graph of $f$. Namely, we need to show that if $\{x_n\}$ is a null sequence in $A$ and $f(x_n) \to y$ in $B$, then $y = 0$.

Pick an ideal $\mathfrak{b} \in S$; note that $\mathfrak{b}$ is closed, by Lemma 9. Put $\mathfrak{a} = f^{-1}(\mathfrak{b})$, so that $\mathfrak{a}$ is closed in $A$, each of $A/\mathfrak{a}$ and $B/\mathfrak{b}$ inherits a quotient norm, and $A/\mathfrak{a} \to B/\mathfrak{b}$ is injective. Since $B/\mathfrak{b}$ is finite dimensional over $K$, so is $A/\mathfrak{a}$, and any linear map between finite dimensional $K$-vector spaces is continuous. (Remember that there is only one equivalence class of norms on a finite dimensional $K$-vector space!) Thus we must have $y \in \mathfrak{b}$ for each $\mathfrak{b}$; since the $\mathfrak{b}$ have trivial intersection, we have $y = 0$. $\square$

**Corollary 11.** *Any $K$-algebra homomorphism between affinoid algebras is continuous. In particular, for a fixed affinoid algebra $A$, the quotient norms induced by different presentations $T_n \to A$ are all equivalent.*

Now I can clarify my remark from earlier about norm-preserving automorphisms of $T_n$.

**Corollary 12.** *Any automorphism of $T_n$ preserves the Gauss norm.*

*Proof.* This follows from the previous corollary and the fact that we can recover the Gauss norm from the topology of $T_n$. Namely, an element $f \in T_n$ satisfies $\|f\| \leq 1$ if and only if for any null sequence $\{c_n\}$ in $K$, $\{c_n f\}$ is null in $T_n$. $\square$

In other words, the Gauss norm is a "canonical" norm for $T_n$. There is an analogous function on an arbitrary affinoid algebra, but it's only a seminorm in general; we'll talk more about it soon.

## Exercises

These exercises are from Section 6 of my preprint "Full faithfulness for overconvergent $F$-isocrystals", which you can grab from the arXiv if you get stuck.

1. Using distinction, prove that the action of $\mathrm{GL}_n(T_n)$ on $n$-tuples which generate the unit ideal is transitive. (Hint: compare to a proof of the analogous statement for polynomials; see [L, Theorem XXI.3.4].)

2. Prove that every finitely generated module over $T_n$ has a finite free resolution. (Hint: this time, see [L, Theorem XXI.3.6].)

3. Prove the analogue of the Quillen-Suslin theorem for $T_n$: every finitely generated projective module over $T_n$ is free. (Hint: use [L, Theorem XXI.2.1] to show that a finite projective is stably free, that is, its direct sum with some free is free. Then compare [L, Theorem XXI.3.6].)