

## 18.727, Topics in Algebraic Geometry (rigid analytic geometry)

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### $p$ -adic uniformization and Shimura curves

I'm going to wrap things up by, in a sense, coming full circle; we're going to discuss  $p$ -adic uniformization, which ties in closely to the Tate curve we constructed at the very beginning of the course.

**References:** I'm pulling a lot of this out of: J.-F. Boutot et H. Carayol, Uniformisation  $p$ -adique des courbes de Shimura: les théorèmes de Čerednik et de Drinfeld, *Astérisque* **196-197**. I will cite this as [BC]. Thanks to Samit for the reference. Also see [FvdP, Section 5.4] for more details on how to take quotients of  $\mathbb{P}^1$  to make “Mumford curves”.

## Quaternion algebras

Can someone suggest a good reference for this?

A few reminders about quaternion algebras: a *quaternion algebra* over a field  $F$  is a central simple algebra of degree 2 over  $F$ , i.e., a division algebra with center  $F$  of dimension 4 as an  $F$ -vector space, i.e., an element of the Brauer group of  $F$  of order 2. (Whether you allow the exceptional case  $M_2(F)$ , the “split quaternion algebra”, is a matter of convention.)

Over a complete discretely valued field, there is a unique “ring of integers” in a quaternion algebra: it is the set of elements whose norm (element times its conjugate,  $a/k/a$  the determinant of the multiplication-by-said-element map) has absolute value  $\leq 1$ . We call this the *maximal order* of the quaternion algebra. Over a number field  $F$ , by a maximal order of a quaternion algebra  $D$  we mean an  $\mathfrak{o}_F$ -subalgebra  $\mathfrak{o}_D$  with  $\mathfrak{o}_D \otimes_{\mathfrak{o}_F} F = D$ . There exists at least one such, and any two are conjugate.

We say a quaternion algebra  $D$  over a number field *splits* at a place  $v$  (a completion, either archimedean or corresponding to a finite prime) if when you complete at  $v$ , you get the split quaternion algebra rather than an honest one. By class field theory, the number of nonsplit places is always even.

## The Drinfeld upper half-plane

Throughout this lecture and the next, let  $K$  be a *discretely valued* complete nonarchimedean field with *finite* residue field  $k = \mathbb{F}_q$ . Let  $\mathbb{C}$  be the completed algebraic closure of  $K$ . Let  $\pi$  be a uniformizer of  $\mathfrak{o}$ . See [BC, Part I] for details.

Define the *Drinfeld upper half-plane*  $\Omega$  to be the analytic subspace of  $\mathbb{P}_{\mathbb{C}}^1$  obtained by removing the  $K$ -rational points; those form a closed subset in the metric topology, so their complement is admissible and this really makes sense as a rigid analytic space.

Note that the group  $\mathrm{PGL}_2(K)$  acts on  $\Omega$ . The action is not free, because you can have fixed points over quadratic extensions of  $K$ .

One may think of  $\mathbb{P}^1(\mathbb{C})$  as the set of  $\mathbb{C}$ -homothety classes of  $K$ -linear maps  $K^2 \rightarrow \mathbb{C}$ ;  $\Omega$  corresponds to those classes represented by injective maps. So you may think of  $\Omega$  as

classifying “rank two  $K$ -lattices inside  $\mathbb{C}$ ”, much as the complex upper half plane classifies rank two  $\mathbb{Z}$ -lattices inside  $\mathbb{C}$ .

There is also a natural formal scheme  $\widehat{\Omega}$  with generic fibre  $\Omega$ ; it is most easily described in terms of the *building* associated to  $\mathrm{PGL}_2(K)$ , which is the following graph. Consider the set of homothety classes of  $\mathfrak{o}$ -lattices in  $K^2$  as the vertices; join two classes by an edge if you can find representing lattices  $L_1, L_2$  with  $\pi L_1 \subset L_2 \subset L_1$ . The edges out of each vertex can be identified with  $\mathbb{P}_{\mathbb{F}_q}^1$ ; if you imagine the graph as a CW-complex, its points correspond to homothety classes of norms on  $K^2$ . (You should be thinking Berkovich here...) Namely, if  $\mathbf{v}_1, \mathbf{v}_2$  is a basis, and I take a point on the edge joining  $\mathfrak{o}\mathbf{v}_1 + \mathfrak{o}\mathbf{v}_2$  with  $\mathfrak{o}\mathbf{v}_1 + \pi\mathfrak{o}\mathbf{v}_2$ , its corresponding norm is

$$\|a_1\mathbf{v}_1 + a_2\mathbf{v}_2\| = \max\{|a_1|, q^t|a_2|\}$$

with  $t \in [0, 1]$ .

Note that one has a natural map from  $\Omega$  to the building: given an element of  $\Omega$  describing an injective map from  $K^2$  to  $\mathbb{C}$ , compose with the norm on  $\mathbb{C}$  to get a norm on  $K^2$ . Call this map  $\lambda$ .

Each vertex  $v$  of the building gives us a way to identify  $\mathbb{P}_K^1$  with the generic fibre of a projective line over  $\mathfrak{o}$ ; call the latter  $\mathbb{P}_v$ . Let  $\widehat{\Omega}_v$  be the formal completion, along the special fibre, of the complement in  $\mathbb{P}_v$  of the rational points on the special fibre; the corresponding rigid space is  $\lambda^{-1}(v)$ .

If  $vw$  is an edge corresponding to a pair of lattices  $L_1, L_2$  with  $\pi L_1 \subset L_2 \subset L_1$ , let  $\mathbb{P}_{vw}$  be the blowup of  $\mathbb{P}_v$  at the point on its special fibre defined by  $L_2$ ; we have a canonical identification  $\mathbb{P}_{vw} \cong \mathbb{P}_{vw}$ . Let  $\widehat{\Omega}_{vw}$  be the formal completion, along the special fibre, of the complement in  $\mathbb{P}_{vw}$  of the *nonsingular* rational points on the special fibre (so leave in the crossing point); the corresponding rigid space is  $\lambda^{-1}(vw)$ . We can now paste the  $\widehat{\Omega}_v$  and  $\widehat{\Omega}_{vw}$  together to get a formal scheme  $\widehat{\Omega}$  with generic fibre  $\Omega$ , and  $\mathrm{PGL}_2(K)$  acts on it.

There are various ways to interpret  $\widehat{\Omega}$  as a “moduli space” (i.e., it represents some natural functors); see [BC, I.4 and I.5].

Aside: I won’t discuss it further here, but there is also a Drinfeld upper half-space of any dimension  $n$ . You get it by taking  $\mathbb{P}^{n+1}(\mathbb{C})$  and removing all  $K$ -rational hyperplanes (not just  $K$ -rational points!). This carries an action of  $\mathrm{PGL}_{n+1}(K)$ ; I think you can also analogize to other linear groups, but I don’t know where anyone has done that.

## Drinfeld’s half-plane and formal $\mathfrak{o}_D$ -modules

Let  $D$  be a quaternion algebra with center  $K$ . Fix a choice of a quadratic extension  $K'$  of  $K$  contained in  $D$ , and let  $\mathfrak{o}'$  be its ring of integers.

We are interested in *special formal  $\mathfrak{o}_D$ -modules* over a  $\mathfrak{o}$ -algebra  $B$ ; such a thing is a formal  $\mathfrak{o}$ -module  $X$  of dimension 2 (in the sense of last time) plus an action  $i : \mathfrak{o}_D \rightarrow \mathrm{End}(X)$  compatible with the  $\mathfrak{o}$ -action, such that  $\mathrm{Lie}(X)$  becomes a free  $B \otimes_{\mathfrak{o}} \mathfrak{o}'$ -module of rank 1. In particular, we are going to insist that these also have height 4 (I think this is as a  $\mathfrak{o}$ -algebra, but I’m not positive).

Anyway, Drinfeld showed that there exists a universal deformation of such a formal module over  $\mathbb{F}_q$  plus some extra data (a “quasi-isogeny of height 0” on the reduction), which lives on  $\widehat{\Omega \otimes_{\mathfrak{o}} \widehat{\mathfrak{o}^{\text{unr}}}}$ . This is kind of a long story, and it’s in the same spirit as the discussion of Lubin-Tate formal groups from last time (even though the formal groups are all two-dimensional, the extra endomorphisms make the situation look like the Lubin-Tate case) so I’m not going to discuss it further here. See [BC, Section 2] for all the gory details.

## Shimura curves and the upper half plane

Things like modular curves are obtained complex analytically by taking the complex upper half plane and quotienting by the action of a discrete group, like  $\text{PGL}_2(\mathbb{Z})$ . If you take a group commensurable with  $\text{PGL}_2(\mathbb{Z})$  (a *congruence subgroup*), you get a classical modular curve; in that case, you have to compactify by adding in the orbits of  $\mathbb{P}_{\mathbb{Q}}^1$  (the *cusps*). However, Shimura noticed you can also take quotients by things like the unit group of a maximal order in a quaternion algebra (the maximal order is  $\mathcal{O}$ , and get algebraic curves over  $\mathbb{Q}$  (not just over  $\mathbb{C}$ ). Moreover, like the modular curves, which are moduli spaces for elliptic curves plus some extra structure, Shimura’s curves are moduli spaces for certain two-dimensional abelian varieties with extra endomorphisms by that maximal order in the quaternion algebra (so-called “false elliptic curves”).

Let me make this more precise before proceeding to the rigid analogue of this description. Let  $\Delta$  be an indefinite quaternion algebra with center  $\mathbb{Q}$ . (“Indefinite” means that  $\Delta \otimes_{\mathbb{Q}} \mathbb{R}$  is split, so is congruent to  $M_2(\mathbb{R})$  rather than the Hamilton quaternions.) For  $R$  a commutative  $\mathbb{Q}$ -algebra, let  $\Delta^*(R)$  denote the group of units in the noncommutative ring  $\Delta \otimes_{\mathbb{Q}} R$ . Let  $\mathbb{A}_f = \mathbb{Z} \otimes_{\mathbb{Q}} \widehat{\mathbb{Z}}$  denote the ring of finite adèles (where  $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$  is the profinite completion of  $\mathbb{Z}$ ), and let  $\mathcal{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$  be the upper and lower half planes in  $\mathbb{C}$ . Then Shimura’s observation is that for any open compact subgroup  $U$  of  $\Delta^*(\mathbb{A}_f)$ , the (left) quotient

$$\Delta^*(\mathbb{Q}) \backslash [\mathcal{H}^{\pm} \times \Delta^*(\mathbb{A}_f) / U]$$

can be naturally identified with the  $\mathbb{C}$ -points of a *projective* algebraic curve  $S_U$  over  $\mathbb{Q}$  (look Ma, no cusps!), and that the curve is a coarse moduli space for a moduli problem concerning two-dimensional abelian varieties with endomorphisms by  $\mathfrak{o}_{\Delta}$  and “level  $U$  structure”. (And of course if  $U$  is “sufficiently small”, the moduli space is even fine.)

## Shimura curves and the rigid upper half plane

Let  $p$  be a prime where  $\Delta$  does not split, so that  $\Delta_p = \Delta \otimes_{\mathbb{Q}} \mathbb{Q}_p$  is an honest quaternion algebra over  $\mathbb{Q}_p$ . Then  $\Delta_p^*$  has a unique maximal compact subgroup  $U_p^0$ , namely the units of the maximal order of  $\Delta_p$ , and there is a decreasing sequence of compact subgroups  $U_p^n$  given by units congruent to 1 modulo  $p^n$ .

Suppose my level structure  $U$  looks like  $U_p^n$  times some compact open subgroup  $U^p$  of the prime-to- $p$  part of  $\Delta^*(\mathbb{A}_f)$ . The theorem of Cherednik-Drinfeld (you can spell that first one

“Čerednik” if you prefer; you can spell that second one “Drinfel’d” if you prefer too) says that if you form the rigid analytic quotient

$$\mathrm{GL}_2(\mathbb{Q}_p) \backslash [(\widehat{\Omega} \widehat{\otimes} \widehat{\mathbb{Z}}_p^{\mathrm{unr}}) \times Z_U],$$

you get the analytification of  $S_U$  again! Here

$$Z_U = U^p \backslash \overline{\Delta}^*(\mathbb{A}_f) / \overline{\Delta}^*(\mathbb{Q})$$

is a double coset space whose quotient by  $\overline{\Delta}^*(\mathbb{Q}_p)$  is finite.

But what the heck is  $\overline{\Delta}$ ? I’ll come back to that in a moment. First, let me point out that (if I read [BC] correctly) Čerednik proved this originally by some not-so-enlightening argument; Drinfeld’s contribution was to match up the moduli interpretation of the Shimura curve with the moduli interpretation of  $\widehat{\Omega}$ , giving a much more conceptual proof of the theorem in the bargain.

## But what the heck is $\overline{\Delta}$ ?

Glad you asked.  $\overline{\Delta}$  is the quaternion algebra obtained from  $\Delta$  by switching the invariants at  $p$  and  $\infty$ ; i.e.,  $\overline{\Delta}$  looks like  $\Delta$  at all finite primes other than  $p$ , but it splits at  $p$  and is definite (nonsplit in the real place).

Why does that come up? By Tate-Honda (see [BC, Proposition III.2]):

- there is a unique isogeny class of two-dimensional abelian varieties over  $\overline{\mathbb{F}}_p$  equipped with an action of  $\mathfrak{o}_\Delta$ ;
- any such abelian variety  $A$  is isogenous to the product of two supersingular elliptic curves;
- the algebra  $\mathrm{End}_{\mathfrak{o}_\Delta}(A) \otimes \mathbb{Q}$  is isomorphic to  $\overline{\Delta}$ .

The fact that there is this “switcheroo” between the quaternion algebra which acts on the abelian surfaces and the quaternion algebra you use in forming the rigid analytic quotient may seem like a trifle or even a nuisance, but it’s actually a wonderful thing! It underlies certain “hidden symmetries” in the theory of automorphic forms, like the Jacquet-Langlands correspondence. A certain geometric realization of this correspondence is a crucial part of Ribet’s proof of Serre’s epsilon conjecture, and in particular that the modularity of elliptic curves implies Fermat’s last theorem. (I have now told you every last thing I know about this. Go talk to David Helm for more information.)

## That’s all, folks!

Thanks for attending the course. Oh, and if you have corrections marked in your notes, I’d love to get some by email so I can assemble a more definitive compilation of the notes.