18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
More on the zeroes of ζ

In this unit, we derive some results about the location of the zeroes of the Riemann zeta function, including a small zero-free region inside the critical strip.

1 Order of an entire function

For $\alpha > 0$, an entire function $f : \mathbb{C} \to \mathbb{C}$ is said to have *order* $\leq \alpha$ if for all $\beta > \alpha$,

$$f(z) = O(\exp |z|^\beta) \qquad (|z| \to \infty).$$

We say f has order α if it has order $\leq \alpha$ but not order $\leq \beta$ for any $\beta < \alpha$.

Lemma 1. *The function*

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

satisfies

$$|\xi(s)| < \exp(C|s|\log|s|) \qquad (|s| \to \infty),$$

and so is of order ≤ 1. (An analogue is true for L-functions, but that is too easy even to give as an exercise.)

Proof. By the functional equation $\xi(s) = \xi(1-s)$, it suffices to check for $|\operatorname{Re}(s)| \geq 1/2$, in which case

$$\left|\frac{1}{2}s(s-1)\pi^{-s/2}\right| < \exp(C_1|s|)$$
$$|\Gamma(s/2)| < \exp(C_2|s|\log|s|)$$

(see exercises for the second estimate). For ζ, we use the integral representation from the first lecture:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty (x - \lfloor x \rfloor)x^{-s-1}\,dx \qquad (\operatorname{Re}(s) > 0).$$

For $\operatorname{Re}(s) \geq 1/2$, the integral is bounded, so $|\zeta(s)| < C_3|s|$. This yields the claim. □

There is a rich theory of integral functions of finite order due to Hadamard (which I believe was introduced originally for the very purpose of studying ζ). The basic idea is to generalize the fact that a polynomial can be written as a product of linear factors (the Fundamental Theorem of Algebra), to write an entire function as a product of one factor for each zero times an exponential.

To do this, one must first control the number of zeroes of f in a disc. There is no harm in assuming that $f(0) \neq 0$, since otherwise we just divide by a suitable power of z. Then recall the following fact from complex analysis.

Theorem 2 (Jensen's formula)**.** *If $f(0) \neq 0$ and f has no zeroes on the circle $|z| = R$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta = \log|f(0)| + \sum_\rho (\log R - \log|\rho|),$$

where ρ runs over the zeroes of f in the disc $|z| < R$ counted with multiplicity.

Proof. Write $f(z) = (z-\rho_1)\cdots(z-\rho_n)g(z)$, where g is nonzero on the disc $|z| \leq R$, and check the equality for each factor individually. For $z - \rho_i$, this is an easy exercise; for g, apply the Cauchy residue formula to the contour integral $\int \log(g(z))\frac{dz}{z}$ around the circle $|z| = R$, then take real parts. ☐

The right side is also

$$\log|f(0)| + \int_0^R \#\{\rho : |\rho| < r\}\frac{dr}{r}.$$

If $\log|f(z)| < r(|z|)$ for some function r, then the left side of Jensen's formula is bounded by $2r(R)$, whereas the right side is at least

$$\log|f(0)| + \log(2)\#\{\rho : |\rho| \leq R/2\}.$$

Consequently, if $r(R) = O(R^\alpha)$, then the number of roots of f in the disc $|\rho| \leq R$ is also $O(R^\alpha)$. Similarly, the fact that $\log|\xi(s)| = O(|s|\log|s|)$ implies that the number of zeroes of ζ with $|\operatorname{Im}(s)| \leq T$ is $O(T\log T)$, which I claimed without proof in the previous unit.

Now let f be entire of order ≤ 1. Let $\rho_1, \rho_2, \dots$ be the zeroes of f sorted so that $|\rho_1| \leq |\rho_2| \leq \cdots$, and put

$$h(z) = \prod_{n=1}^\infty (1 - z/\rho_n)e^{z/\rho_n}$$

Note that this converges uniformly on any disc, because the multiplicand is

$$1 + \frac{1}{2}\left(\frac{z}{\rho_n}\right)^2 + O\left(\left(\frac{z}{\rho_n}\right)^3\right)$$

and the fact that the number of roots of norm $\leq R$ is $O(R^{1+\epsilon})$ implies that $\sum 1/\rho_n^2$ converges (by partial summation). By a somewhat intricate argument (see Davenport §11 or Ahlfors), it can be shown that f/h is also of order ≤ 1. Since f/h has no zeroes, the function $g(z) = \log(f(z)/h(z))$ is entire and satisfies $|g(z)| = O(|z|^{1+\epsilon})$. Consequently,

$$g_2(z) = \frac{g(z) - g(0) - g'(0)z}{z^2}$$

is entire and bounded, hence constant by Liouville's theorem. This yields the following.

Theorem 3 (Hadamard)**.** *Let $f(z)$ be an entire function of order ≤ 1. Then*

$$f(z) = e^{A+Bz}\prod_{n=1}^\infty (1 - z/\rho_n)e^{z/\rho_n}$$

for some constants A, B.

2 A zero-free region for ζ

We now use the product representation for ξ to obtain a zero-free region for ζ. The idea (due to de la Vallée Poussin (1899)) is to squeeze a bit of extra information out of the proof we used for nonvanishing on the line $\operatorname{Re}(s) = 1$. One way to phrase that argument: since

$$\operatorname{Re}(\log(\zeta(s)) = \sum_p \sum_{n=1}^{\infty} \frac{1}{n} \cos(\operatorname{Im}(s) \log p^n) p^{-n \operatorname{Re}(s)}$$

and

$$3 + 4\cos\theta + \cos 2\theta \geq 0,$$

we have

$$3\operatorname{Re}(\log\zeta(\sigma)) + 4\operatorname{Re}(\log\zeta(\sigma+it)) + \operatorname{Re}(\log\zeta(\sigma+2it)) \geq 0 \qquad (\sigma > 1, t \in \mathbb{R})$$

whereas if $\zeta(1+it)$ vanished, then the sum would tend to $-\infty$ as $\sigma \to 1^+$ (because $4 > 3$).

We can apply the same argument with $\log\zeta$ replaced by its negative derivative

$$-\operatorname{Re}\zeta'(s)/\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-\operatorname{Re}(s)} \cos(\operatorname{Im}(s)\log n)$$

to obtain an analogous inequality

$$-3\operatorname{Re}\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\operatorname{Re}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - \operatorname{Re}\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \geq 0 \qquad (\sigma > 1, t \in \mathbb{R}). \tag{1}$$

Let's see how to use (1) to get some information about zeroes just past the line $\operatorname{Re}(s) = 1$. We do this by bounding above each term on the left side of (1) for σ slightly bigger than 1. For starters, since ζ has a simple pole at $s = 1$,

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1} + *$$

where every $*$ in this argument is a positive constant, but no two need be the same.

Applying Hadamard's theorem and taking a logarithmic derivative, we get

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Adjusting to get rid of the gamma factors, we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - B - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'((s+1)/2)}{\Gamma((s+1)/2)} - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

For $1 \leq \mathrm{Re}(s) \leq 2$ and $|\mathrm{Im}(s)| \geq 1$, everything on the right side aside from the sum over ρ is dominated by $* \log |\mathrm{Im}(s)|$. Hence taking real parts, we obtain

$$-\mathrm{Re}\,\frac{\zeta'(s)}{\zeta(s)} < * \log |\mathrm{Im}(s)| - \sum_\rho \mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Since $\mathrm{Re}(\rho) > 0$ and $\mathrm{Re}(s-\rho) > 0$, we also have $\mathrm{Re}(1/\rho) > 0$ and $\mathrm{Re}(1/(s-\rho)) > 0$, so the sum over ρ is positive. Hence

$$-\mathrm{Re}\,\frac{\zeta'(s)}{\zeta(s)} < * \log |\mathrm{Im}(s)|;$$

this is the estimate I'll use for $s = \sigma + 2it$.

Let t be the imaginary part of a zero ρ of ζ; I will bound $-\mathrm{Re}\,\frac{\zeta'(s)}{\zeta(s)}$ for $s = \sigma + it$ by keeping only the summand corresponding to ρ. Namely, if $\rho = \beta + it$, then I get

$$-\mathrm{Re}\,\frac{\zeta'(s)}{\zeta(s)} < * \log |t| - \frac{1}{\sigma - \beta}.$$

From (1), I now deduce

$$\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + * \log |t|.$$

For $\sigma = 1 + */(\log |t|)$, I can deduce

$$\beta < 1 - \frac{*}{\log |t|}.$$

In other words:

Theorem 4. *There exists a constant $c > 0$ such that there is no zero of ζ in the region* $\mathrm{Re}(s) \geq 1 - c/\log \mathrm{Im}(s), \mathrm{Im}(s) \geq 1$.

By von Mangoldt's formula (presented in the previous unit, with proof still to follow), this yields a nontrivial error bound in the prime number theorem, namely

$$\pi(x) = \mathrm{li}(x) + O(x \exp(-c\sqrt{\log x}))$$

(exercise).

3 What about L-functions?

The previous argument goes through more or less unchanged for L-functions. But there is a new complication: remember that we only looked at zeroes whose imaginary part was not too small. We took $|\mathrm{Im}(s)| \geq 1$, but the lower bound could have been any *fixed* positive constant.

The real issue is that while we can check once and for all that $\zeta(s)$ has no zeroes on the real line, we cannot rule this out for L-functions. But $L(s, \chi)$ could in principle have a real zero; such a hypothetical zero is called a *Siegel zero*. These can only occur for real nonprincipal characters.

Exercises

1. Prove that $1/\Gamma$ is entire of order ≤ 1. Then prove that

$$\frac{1}{s\Gamma(s)} = e^{\gamma s} \prod_{n=1}^{\infty} (1 + s/n) e^{-s/n} \qquad (s \neq 0, -1, -2, \dots),$$

   where γ is Euler's constant, by applying Hadamard's theorem.

2. Prove that
$$\frac{\Gamma'(s)}{\Gamma(s)} = \log(s) + O(|s|^{-1}) \qquad (|s| \to \infty, \mathrm{Re}(s) \geq 1/2).$$
   (Hint: use the previous exercise.)

3. Derive the estimate

$$|\Gamma(s/2)| < \exp(C_2 |s| \log|s|) \qquad (\mathrm{Re}(s) \geq 1/2)$$

   by first proving a suitably strong version of Stirling's formula, e.g.,

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}) \qquad (|s| \to \infty, \mathrm{Re}(s) \geq 1/2).$$

4. Prove that a function of order $\leq \alpha$ need not satisfy $|f(z)| = O(\exp(|z|^{\alpha}))$. (Hint: look at ζ on the positive real axis.)

5. Find the constants A and B in the product representation for ξ given by Hadamard's theorem. Then deduce as a corollary that $\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi$.

6. Use the zero-free region and von Mangoldt's formula to prove that for some $c > 0$,

$$\pi(x) = \mathrm{li}(x) + O(x \exp(-c\sqrt{\log x})).$$

   (By contrast, the leading term is $x \exp(-\log\log x)$.)