**18.786 supplement: linear disjointness**

Let me try to alleviate a bit of the confusion about the concept of linear disjointness (thanks to Dani Kane for helping straighten some of this out). This corrects some assertions from the March 14 lecture.

Let $L_1, L_2$ be finite extensions of a field $K$. We say that $L_1$ and $L_2$ are *linearly disjoint* if the $K$-algebra $L_1 \otimes_K L_2$ is a field (note that it is enough for it to be an integral domain, since we proved in class that any integral domain which is integral over a field is also a field). If so, it is then isomorphic to the compositum $L_1L_2$ within any overfield $L_3$ containing both $L_1$ and $L_2$ (because the multiplication map from $L_1 \otimes_K L_2$ to $L_3$ will be injective and its image will be a field containing both $L_1$ and $L_2$, so $L_1L_2$ can be no larger).

It is not true in general that just knowing that $L_1 \cap L_2 = K$ inside some overfield $L_3$ is enough to say that $L_1$ and $L_2$ are linearly disjoint. (For instance, look at two different copies of $\mathbb{Q}[x]/(x^3 - 2)$ inside $\mathbb{C}$; their compositum has degree 6, not 9, over $\mathbb{Q}$.)

However, if $L_1$ and $L_2$ are *Galois* over $K$, then it is true that the equality $L_1 \cap L_2 = K$ in any overfield implies that $L_1$ and $L_2$ are linearly disjoint. Proof: we first check that the compositum of two Galois extensions is Galois (without any extra hypothesis). Let $L_3$ be the compositum in some overfield. Since $L_1/K$ is separable, it is generated by a root of a separable polynomial; that root also generates $L_3$ over $L_2$, so $L_3/L_2$ is separable. Since $L_3/L_2$ and $L_2/K$ are separable, so is $L_3/K$. Since $L_1$ and $L_2$ are Galois, they are normal: they are splitting fields for some polynomials $P_1$ and $P_2$ over $K$. Then $L_3$ is a splitting field for $P_1P_2$, so it is also normal over $K$. Since $L_3/K$ is normal and separable, it is Galois (but this, or more precisely the fact that normal plus separable implies that the automorphism group is as big as the extension degree, is nontrivial Galois theory!).

Put $G_i = \mathrm{Gal}(L_i/K)$; then $G_3$ surjects onto $G_1$ and $G_2$ via the Galois correspondence. Let $H_i$ be the kernel of $G_3 \to G_i$ for $i = 1, 2$; then $H_1 \cap H_2$ fixes both $L_1$ and $L_2$, and since $L_3$ is the compositum of those, $H_1 \cap H_2$ must fix $L_3$. Hence $H_1 \cap H_2 = \{e\}$.

Finally, the hypothesis $L_1 \cap L_2 = K$ implies, via the Galois correspondence, that there is no normal subgroup of $G_3$ containing both $H_1$ and $H_2$; that is, $H_1H_2 = G_3$. We have maps $G_3 \to G_1$ and $G_3 \to G_2$ giving a map $G_3 \to G_1 \times G_2$; by elementary group theory, this map must now be an isomorphism. In particular, $[L_3 : K] = [L_1 : K][L_2 : K]$, so the surjection $L_1 \otimes_K L_2 \to L_3$ must be an isomorphism.