p-adic differential equations
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Absolute values

In this unit, we recall some basic facts about absolute values, primarily of the nonarchimedean sort; the treatment is not at all comprehensive, as it is only meant as a review. A couple of proofs will forward reference the unit on Newton polygons.

1 Absolute values on fields

Let us start by recalling some basic definitions from analysis, without yet specializing to the nonarchimedean case.

An *absolute value* (or *norm*) on a field F is a function $|\cdot| : F \to [0, +\infty)$ satisfying the following conditions.

(a) For $f \in F$, $|f| = 0$ if and only if $f = 0$.

(b) For $f, g \in F$, $|f + g| \leq |f| + |g|$.

(c) For $f, g \in F$, $|fg| = |f||g|$.

An inclusion $F \hookrightarrow F'$ of fields equipped with absolute values is *isometric* if the restriction to F' of the absolute value on F' coincides with the absolute value on F.

A *Cauchy sequence* in F under $|\cdot|$ is a sequence $\{x_n\}_{n=0}^{\infty}$ such that for any $\epsilon > 0$, there exists an integer N such that for all integers $m, n \geq N$, $|x_m - x_n| < \epsilon$. We say the sequence $\{x_n\}_{n=0}^{\infty}$ is *convergent* if there exists $x \in F$ such that for any $\epsilon > 0$, there exists an integer N such that for all integers $n \geq N$, $|x - x_n| < \epsilon$; in this case, the sequence is automatically Cauchy, x is uniquely determined (because of property (a)), and we call x the *limit* of the sequence. We say F is *complete* under $|\cdot|$ if every Cauchy sequence is convergent.

Theorem 1. *Let F be a field equipped with an absolute value $|\cdot|$. Then there exists a field F' equipped with an absolute value $|\cdot|'$ under which it is complete, and an isometric injection $F \to F'$ with dense image for the metric topology. Moreover, the construction is functorial in F.*

We call F' the *completion* of F.

Proof. Take F' to be the set of equivalence classes of Cauchy sequences, where two Cauchy sequences are equivalent if any sequence obtained by merging them is also Cauchy; note that it does not matter how the merge is done, as as the property of being a Cauchy sequence is invariant under rearranging terms. For more details, see any basic analysis text. □

We say two absolute values $|\cdot|, |\cdot|'$ are *metrically equivalent* if they induce the same metric topology on F.

Proposition 2. *Two absolute values $|\cdot|, |\cdot|'$ on a field F are metrically equivalent if and only if there exists $c > 0$ such that $|x|' = |x|^c$ for all $x \in F$.*

Proof. See [DGS94, Lemma I.1.2]. □

2 Valuations and nonarchimedean absolute values

A *real valuation* on a field F is a function $v : F \to \mathbb{R} \cup \{+\infty\}$ with the following properties.

(a) For $f \in F$, $v(f) = +\infty$ if and only if $f = 0$.

(b) For $f, g \in F$, $v(f + g) \geq \min\{v(f), v(g)\}$.

(c) For $f, g \in F$, $v(fg) = v(f) + v(g)$.

If v is a valuation, then $|\cdot| = e^{-v(\cdot)}$ is an absolute value on F which is *nonarchimedean*, meaning that it satisfies the strong triangle inequality

$$|f + g| \leq \max\{|f|, |g|\} \qquad (f, g \in F).$$

Conversely, if $|\cdot|$ is a nonarchimedean absolute value, then $v(\cdot) = -\log|\cdot|$ is a real valuation. It is worth noting that there are comparatively few archimedean (not nonarchimedean) absolute values.

Theorem 3 (Ostrowski)**.** *Let F be a field equipped with an absolute value $|\cdot|$. Then $|\cdot|$ fails to be nonarchimedean if and only if the sequence $|1|, |2|, |3|, \ldots$ is unbounded. In that case, F is isomorphic to a subfield of $\mathbb{C}$ with the induced absolute value.*

Proof. Exercise, or see [Rbe00, §2.1.6] and [Rbe00, §2.2.4] respectively. □

A field equipped with a nonarchimedean absolute value will be referred to simply as a *nonarchimedean field.* The term *ultrametric field* is also used, with the term "ultrametric" referring to a metric satisfying the strong triangle inequality.

The *value group* of a nonarchimedean field F is the image of $F^\times$ under $|\cdot|$, viewed as a subgroup of $\mathbb{R}^+$. If this subgroup is discrete, we say F is *discretely valued.*

If F is equipped with a valuation, we define

$$\begin{aligned} \mathfrak{o}_F &= \{f \in F : v(f) \geq 0\} \\ \mathfrak{m}_F &= \{f \in F : v(f) > 0\} \\ \kappa_F &= \mathfrak{o}_F/\mathfrak{m}_F. \end{aligned}$$

Note that $\mathfrak{o}_F$ is a local ring (the *valuation ring* of F), $\mathfrak{m}_F$ is the maximal ideal of $\mathfrak{o}_F$, and κ_F is a field (the *residue field* of F).

3 Norms on vector spaces

Let F be a field equipped with an absolute value $|\cdot|$. Let V be a vector space over F. A *norm on V* compatible with $|\cdot|$ is a function $|\cdot|_V : V \to [0, +\infty)$ with the following properties.

(a) For $x \in V$, $|x|_V = 0$ if and only if $x = 0$.

(b) For $x, y \in V$, $|x+y|_V \leq |x|_V + |y|_V$.

(c) For $f \in F$, $x \in V$, $|fx|_V = |f||x|_V$.

If F is nonarchimedean, we conventionally require that $|\cdot|_V$ should also satisfy the strong triangle inequality

$$|x+y|_V \leq \max\{|x|_V, |y|_V\} \qquad (x, y \in V).$$

This is a real extra restriction; see the exercises.

We say two norms $|\cdot|_V, |\cdot|'_V$ are *equivalent* if there exist constants $c_1, c_2 > 0$ such that

$$|x|_V \leq c_1 |x|'_V, \qquad |x|'_V \leq c_2 |x|_V \qquad (x \in V).$$

Theorem 4. *If F is complete under $|\cdot|$, then any two norms on a finite-dimensional F-vector space compatible with $|\cdot|$ are equivalent to each other.*

In particular, this is true of the supremum norm defined by any basis $e_1, \dots, e_n$:

$$|c_1 e_1 + \cdots + c_n e_n|_V = \max_i \{|c_i|\};$$

this ensures that there exists at least one norm on V.

Proof. In the nonarchimedean case, see [DGS94, Theorem I.3.2]. Otherwise, apply Theorem 3 to deduce that $F = \mathbb{R}$ or $F = \mathbb{C}$, then use compactness of the unit ball. ☐

4 Examples of nonarchimedean absolute values

For any field F, there is a *trivial absolute value* of F defined by

$$|f|_{\mathrm{triv}} = \begin{cases} 1 & f \neq 0 \\ 0 & f = 0. \end{cases}$$

This absolute value is nonarchimedean, and F is complete under it. The trivial case will always be allowed unless explicitly excluded; it is often a useful input into a highly nontrivial construction, as in the next few examples.

Let F be any field, and let $F((t))$ denote the field of formal Laurent series. The *t-adic valuation* v_t on F is defined as follows: for $f = \sum_i c_i t^i \in F((t))$, $v_t(f)$ is the least i for which $c_i \neq 0$. This exponentiates to give a *t-adic absolute value*, under which $F((t))$ is complete and discretely valued.

For F a nonarchimedean field and $\rho > 0$, the *ρ-Gauss norm* (or the *(t,ρ)-Gauss norm*, in case we need to specify t explicitly) on the rational function field $F(t)$ is defined as follows: for $f = P/Q$ with $P, Q \in F[t]$, write $P = \sum_i P_i t^i$ and $Q = \sum_j Q_j t^j$, and put

$$|f|_\rho = \max_i \{|P_i| \rho^i\} / \max_j \{|Q_j| \rho^j\}.$$

Note that $F(t)$ is discretely valued under the ρ-Gauss norm if and only if either:

(a) F carries the trivial absolute value, in which case the norm is equivalent to the t-adic absolute value no matter what ρ is; or

(b) F carries a nontrivial absolute value, and ρ belongs to the divisible closure of the value group of F.

So far we have not mentioned the principal examples from number theory; let us do so now. For p a prime number, the *p-adic absolute value* $|\cdot|_p$ on $\mathbb{Q}$ is defined as follows: given $f = r/s$ with $r, s \in \mathbb{Z}$, write $r = p^a m$ and $s = p^b n$ with m, n not divisible by p, then put

$$|f|_p = p^{-a+b}.$$

In particular, we have normalized so that $|p| = p^{-1}$; this convention is usually taken so as to make the *product formula* hold. Namely, for any $f \in \mathbb{Q}$, if $|\cdot|_\infty$ denotes the usual archimedean absolute value, then

$$|f|_\infty \prod_p |f|_p = 1.$$

Completing $\mathbb{Q}$ under $|\cdot|_p$ gives the *field of p-adic numbers* $\mathbb{Q}_p$; it is discretely valued. Its valuation subring is denoted $\mathbb{Z}_p$ and called the *ring of p-adic integers*.

Theorem 5 (Ostrowski)**.** *Any nontrivial nonarchimedean absolute value on $\mathbb{Q}$ is equivalent to the p-adic absolute value for some prime p.*

Proof. See [Rbe00, §2.2.4]. □

To equip extensions of $\mathbb{Q}_p$ with absolute values, we use the following result.

Theorem 6. *Let F be a field complete for a nonarchimedean absolute value $|\cdot|$. Then any finite extension E of F admits a unique extension of $|\cdot|$ to an absolute value on E.*

Proof. We only prove uniqueness now; existence will be established in the unit on Newton polygons. Let $|\cdot|_1$ and $|\cdot|_2$ be two extensions of $|\cdot|$ to absolute values on E. Then these in particular give norms on E viewed as an F-vector space; by Theorem 4, these norms are equivalent. That is, there exist $c_1, c_2 > 0$ such that

$$|x|_1 \leq c_1|x|_2, \qquad |x|_2 \leq c_2|x|_1 \qquad (x \in E).$$

We now use the extra information that $|\cdot|_1$ and $|\cdot|_2$ are multiplicative, because they really are norms on E as a field in its own right. That is, for any positive integer n, we may substitute x^n in place of x in the previous inequalities, then take n-th roots, to obtain

$$|x|_1 \leq c_1^{1/n}|x|_2, \qquad |x|_2 \leq c_2^{1/n}|x|_1 \qquad (x \in E).$$

Letting $n \to \infty$ gives $|x|_1 = |x|_2$, as desired. □

Don't forget that the completeness of F is crucial. For instance, the 5-adic absolute value on $\mathbb{Q}$ extends in two different ways to the Gaussian rational numbers $\mathbb{Q}(i)$, depending on whether you want $|2+i| = 5^{-1}, |2-i| = 1$ or vice versa.

Because of the uniqueness in Theorem 6, it also follows that any algebraic extension E of F, finite or not, inherits a unique extension of $|\cdot|$. However, if $[E : F] = \infty$, then E is not complete, so we may prefer to use its completion instead. For instance, if $F = \mathbb{Q}_p$, we define $\mathbb{C}_p$ to be the completion of an algebraic closure of $\mathbb{Q}_p$. You might worry that this may launch us into an endless cycle of completion and algebraic closure, but fortunately this does not occur.

Theorem 7. *Let F be an algebraically closed field equipped with a nonarchimedean absolute value. Then the completion of F is also algebraically closed.*

Proof. See the unit on Newton polygons. For the case of the completed algebraic closure of $\mathbb{Q}_p$, see also [Rbe00, §3.3.3]. $\square$

5 Spherical completeness

For nonarchimedean fields, there is an important distinction between two different notions of completeness, which does not appear in the archimedean case.

A metric space is *complete* if any decreasing sequence of closed balls with radii tending to 0 has nonempty intersection; this is a reformulation of the Cauchy sequence definition. A metric space is *spherically complete* if any decreasing sequence of closed balls, regardless of radii, has nonempty intersection. (For a topological vector space, the term *linearly compact* is also used.)

The fields $\mathbb{R}$ and $\mathbb{C}$ with their usual absolute value are spherically complete. Also, any complete nonarchimedean field which is discretely valued, e.g., $\mathbb{Q}_p$ or $\mathbb{C}((t))$, is spherically complete. However, any infinite algebraic extension of $\mathbb{Q}_p$ is not spherically complete.

Theorem 8 (Kaplansky-Krull)**.** *Any nonarchimedean field embeds isometrically into a spherically complete nonarchimedean field. (However, the construction is not functorial.)*

Proof. Since completion is functorial, we may assume we are starting with a complete nonarchimedean field. It was originally shown by Krull [Kru32, Theorem 24] that any complete nonarchimedean field admits an extension which is *maximally complete*, in the sense of not admitting any extensions preserving both the value group and the residue field. (In fact, this is not difficult to prove using Zorn's lemma.) The equivalence of this condition with spherical completeness was then proved by Kaplansky [Kap42, Theorem 4].

One can also prove the result more directly; for instance, the case of $\mathbb{Q}_p$ is explained in detail in [Rob00, §3]. $\square$

6 Notes

The condition of spherical completeness is quite important in nonarchimedean functional analysis, as it is needed for the Hahn-Banach theorem to hold. (By contrast, the nonarchimedean version of the open mapping theorem requires only completeness of the field.) For expansion of this remark, we recommend [Sch02]; an older reference is [vR78].

Although the construction of the spherical completion of a nonarchimedean field is not functorial, it is possible to make a canonical construction using generalized power series (Mal'cev-Neumann series); this was described by Poonen [Poo93].

7 Exercises

1. Prove Ostrowski's theorem (Theorem 3).

2. Exhibit an example to show that even for a finite-dimensional vector space V over a complete nonarchimedean field F, the requirement that a norm on $|\cdot|_V$ must satisfy the strong triangle inequality is not superfluous. (That is, a function $|\cdot|_V : V \to [0, \infty)$ can satisfy the ordinary triangle inequality plus conditions (a) and (c) without satisfying the strong triangle inequality.)

3. Prove that the valuation ring $\mathfrak{o}_F$ of a nonarchimedean field is noetherian if and only if F is discretely valued.

4. Use Theorem 6 to prove that for any field F, any nonarchimedean absolute value $|\cdot|$ on F, and any finite extension of E, there exists *at least one* extension of $|\cdot|$ to an absolute value on E.

5. Here is a more exotic variation of the t-adic valuation. Choose $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. First prove that on the rational function field $F(t_1, \dots, t_n)$, there is a valuation v_α such that $v(f) = 0$ for all $f \in F^*$ and $v(t_i) = \alpha_i$ for $i = 1, \dots, n$. Then prove that if $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$, the valuation v_α is uniquely determined by these specification. (Optional: prove that this is the only case where v_α is unique.)