p-adic differential equations
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Convexity and monotonicity for subsidiary radii

In this unit, we prove some theorems governing the variation of the subsidiary radii of a differential module on a disc or annulus.

1 Setup

Let M be a finite free differential module of rank n over $K\langle \alpha/t, t/\beta \rangle$, where $0 \leq \alpha \leq \beta$. We are interested in the variation of the subsidiary radii of $M \otimes F_\rho$ as ρ ranges over $[\alpha, \beta]$.

The properties we are interested in are more convenient to describe in logarithmic terms, so we set notation as follows. For $\rho \in [\alpha, \beta]$, let $R_1(\rho), \dots, R_n(\rho)$ be the extrinsic subsidiary radii of $M \otimes F_\rho$ in increasing order, so that $R_1(\rho) = R(M \otimes F_\rho)$ is the generic radius of convergence of $M \otimes F_\rho$. For $r \in [-\log \beta, -\log \alpha]$, define

$$f_i(r) = -\log R_i(e^{-r}),$$

so that $f_i(r) \geq r$ for all r. We will write $f_i(M, r)$ instead of $f_i(r)$ in case there is ambiguity about which M we are considering.

2 Main results

We now state the main results of this unit.

Theorem 1. *Let M be a finite free differential module of rank n over $K\langle \alpha/t, t/\beta \rangle$.*

(a) (Linearity) For $i = 1, \dots, n$, the functions f_i are continuous and piecewise affine.

(b) (Integrality) If $i = n$ or $f_i(r_0) > f_{i+1}(r_0)$, then the slopes of $f_1 + \cdots + f_i$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each f_i belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}$.

(c) (Convexity) For $i = 1, \dots, n$, the function $f_1 + \cdots + f_i$ is convex.

(d) (Monotonicity) Suppose that $\alpha = 0$. For $i = 1, \dots, n$, for any point r_0 where $f_i(r_0) > r_0$, the slopes of $f_1 + \cdots + f_i$ are nonpositive in some neighborhood of r_0. (Remember that $f_i(r) = r$ for r sufficiently large.)

Note that we have integrality for the slopes of the f_i but not for the values. Using the integrality result for intrinsic generic radii proved in a previous unit, we can at least deduce the following. (One can prove something similar for subsidiary radii, but the statement is a bit more complicated.)

Theorem 2. *Set notation as in Theorem 1. Let h be a nonnegative integer and pick $m \in \{1, \dots, n\}$. Suppose that the following hold for some r_0:*

(a) $f_m(r_0) = f_1(r_0)$;

(b) either $m = n$, or $f_m(r_0) > f_{m+1}(r_0)$;

(c) $f_1(r_0) - r_0 > \frac{p^{-h}}{p-1} \log p$.

Then for r in some neighborhood of r_0,

$$f_1(r) - r \in \frac{1}{mp^h} v(K^\times) + \frac{1}{m} r\mathbb{Z}.$$

In fact, one cannot do better than this; see §4.

3 Variation of Newton polygons

The formulation of Theorem 1 is motivated by the following result, which will also be used in the proof. (Note that because of a sign discrepancy, convexity is traded for concavity in (c), and nonpositive slopes are traded for nonnegative slopes in (d).)

Theorem 3. *Let $P \in K\langle \alpha/t, t/\beta \rangle[T]$ be a polynomial of degree n. For $r \in [-\log \beta, -\log \alpha]$, let $f_1(r), \dots, f_n(r)$ be the slopes of the Newton polygon of P under $|\cdot|_{e^{-r}}$, in increasing order.*

(a) (Linearity) For $i = 1, \dots, n$, the functions f_i are continuous and piecewise affine.

(b) (Integrality) If $i = n$ or $f_i(r_0) < f_{i+1}(r_0)$, then the slopes of $f_1 + \cdots + f_i$ in some neighborhood of r belong to $\mathbb{Z}$. Consequently, the slopes of each f_i belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}$.

(c) (Concavity) Suppose that P is monic. For $i = 1, \dots, n$, the function $f_1 + \cdots + f_i$ is concave.

(d) (Monotonicity) Suppose that P is monic and that $\alpha = 0$. For $i = 1, \dots, n$, the slope of $f_1 + \cdots + f_i$ is nonnegative.

Proof. Write $P = \sum_{i=0}^n P_i T^i$ with $P_i \in K\langle \alpha/t, t/\beta \rangle$. Write $\mathrm{NP}_r(P)$ for the Newton polygon of P measured with respect to $v_r(\cdot) = -\log |\cdot|_{e^{-r}}$.

For $s \in \mathbb{R}$ and $r \in [-\log \beta, -\log \alpha]$, put

$$v_{s,r}(P) = \min_i \{v_r(P_i) + is\};$$

that is, $v_{s,r}(P)$ is the y-intercept of the supporting line of $\mathrm{NP}_r(P)$ of slope s.

The function $v_r(P_i)$ is continuous in r and piecewise affine with slopes in $\mathbb{Z}$; by the Hadamard three circles lemma, it is also concave. Since $v_{s,r}(P)$ is the minimum of finitely many continuous, piecewise affine, concave functions of r with slopes in $\mathbb{Z}$, so then is $v_{s,r}(P)$.

Note also that $v_{s,r}(P)$ is concave as a function of the pair (r, s), since each function $(r, s) \mapsto v_r(P_i) + is$ has that property.

Note that $f_1(r) + \cdots + f_i(r)$ is the difference between the y-coordinates of the points of $\mathrm{NP}_r(P)$ of x-coordinates $i - n$ and $-n$. That is,

$$f_1(r) + \cdots + f_i(r) = \sup_s \{v_{s,r}(P) - (n-i)s\} - v_r(P_n). \tag{1}$$

Moreover, the supremum in (1) is achieved by some s whose denominator is bounded by n. Consequently, $f_1(r) + \cdots + f_i(r)$ is continuous and piecewise affine, proving (a).

If $i = n$ or $f_i(r_0) > f_{i+1}(r_0)$, then the point of $\mathrm{NP}_{r_0}(P)$ of x-coordinate $i - n$ is a vertex, and likewise for r in some neighborhood of r_0. In that case, for r near r_0,

$$f_1(r) + \cdots + f_i(r) = v_r(P_{n-i}) - v_r(P_n),$$

proving (b).

Assume hereafter that P is monic, so that $P_n = 1$ and (1) reduces to

$$f_1(r) + \cdots + f_i(r) = \sup_s \{v_{s,r}(P) - (n-i)s\}.$$

It is not immediately clear from this that $f_1 + \cdots + f_r$ is concave, since we are taking the supremum rather than the infimum of a collection of concave functions. To get around this, pick $r_1, r_2 \in [-\log \beta, -\log \alpha]$ and put $r_3 = ur_1 + (1-u)r_2$ for some $u \in [0, 1]$. For $j \in \{1, 2\}$, choose s_j achieving the supremum in (1) for $r = r_j$. Put $s_3 = us_1 + (1-u)s_2$; then using the convexity of $v_{s,r}(P)$ in both s and r, we have

$$\begin{aligned}
f_1(r_3) + \cdots + f_i(r_3) &\geq v_{s_3,r_3}(P) - (n-i)s_3 \\
&\geq u(v_{s_1,r_1}(P) - (n-i)s_1) + (1-u)(v_{s_2,r_2}(P) - (n-i)s_2) \\
&= u(f_1(r_1) + \cdots + f_i(r_1)) + (1-u)(f_1(r_2) + \cdots + f_i(r_2)).
\end{aligned}$$

This yields concavity for $f_1 + \cdots + f_i$, proving (c).

Assume finally that $\alpha = 0$ (and P is still monic). Then each $v_r(P_i)$ is a nondecreasing function of r, as then is each $v_{s,r}(P)$. Since $v_r(P_n) = 0$, $f_1 + \cdots + f_r$ is nondecreasing by (1), proving (d). $\square$

4 Key differences

Having drawn an analogy between our original theorem and Theorem 3, we must now indicate some respects in which the analogy falls short.

Besides having everything negated (flipping concavity to convexity), our target theorem also has a boundary case that does not occur in the Newton polygon case. That is because subsidiary radii can "max out" by achieving equality in the bound $f_i(r) \geq r$, at which one has an abrupt change of behavior which undermines the nonpositivity property for slopes.

Another important difference must be noted between Theorem 3 and our desired results. In Theorem 3, the function $f_1 + \cdots + f_n$ is piecewise of the form $mr + b$ where $m \in \mathbb{Z}$, but b is also constrained: it must belong to the additive value group of K. By contrast, this need not be the case in Theorem 1 or 1, as demonstrated by the following example. (See Theorem 2 for the best possible affirmative result.)

Pick $\lambda \in K^\times$ and $0 < \alpha \leq \beta$ such that for $\rho \in [\alpha, \beta]$,

$$p^{1/(p-1)} < |\lambda|\rho^{-p} < p^{p/(p-1)}.$$

Let M be the differential module over $K\langle \alpha/t, t/\beta\rangle$ generated by v satisfying $D(v) = -p\pi\lambda t^{-p-1}$. For $\rho \in [\alpha, \beta]$, using the supremum norm on $M \otimes F_\rho$ given by w, we compute

$$|D|_{M\otimes F_\rho} = p^{-p/(p-1)}|\lambda|\rho^{-p-1} < \rho^{-1};$$

this tells us that $|D|_{\mathrm{tsp},M\otimes F_\rho} \leq \rho^{-1}$ but nothing stronger.

To compute $R(M \otimes F_\rho)$, we construct the module M' over $K\langle \alpha^p/t^p, t^p/\beta^p\rangle$ with generator w and $D'(w) = -\pi\lambda(t^p)^{-2}$. In this case, we read off

$$|D'|_{M'\otimes F'_\rho} = p^{-1/(p-1)}|\lambda|\rho^{-2p} > \rho^{-p}$$

so this also computes $|D'|_{\mathrm{tsp},M'\otimes F'_\rho}$. We thus have

$$R(M' \otimes F'_\rho) = |\lambda|^{-1}\rho^{2p}$$
$$R(M \otimes F_\rho) = |\lambda|^{-1/p}\rho^2,$$

where the latter holds by the Frobenius antecedent theorem because $M = \phi^*(M')$. (More precisely, we first find that $R(M \otimes F_\rho) \geq |\lambda|^{-1/p}\rho^2 > p^{-1/(p-1)}\rho$, so M has a Frobenius antecedent; we then note that $R(M' \otimes F'_\rho) > p^{-p/(p-1)}\rho^p$, so the Frobenius antecedent of M is forced to equal M'. We then get $R(M \otimes F_\rho) = |\lambda|^{-1/p}\rho^2$.)

In particular,

$$f_1(r) = 2r + \frac{1}{p}\log|\lambda|$$

has constant term which need not belong to the value group of K.

5 Convexity of the generic radius

As a prelude to tackling Theorem 1, we give a quick proof of convexity of the function f_1, corresponding to the generic radius of convergence. This argument applies to both discs and annuli, and can be (and historically was) used in place of the full strength of Theorem 1 for many purposes.

Choose a basis of M, and let D_s be the basis via which D^s acts on M. Then recall that

$$R_1(\rho) = \min\{\rho, p^{-1/(p-1)} \liminf_{s\to\infty} |D_s|_\rho^{-1/s}\}.$$

For each s, the function $r \mapsto -\log |D_s|_{e^{-r}}^{-1/s}$ is convex in r by the Hadamard three circles lemma. This implies the convexity of

$$f_1(r) = \max\{r, \frac{1}{p-1}\log p + \limsup_{s\to\infty}(-\log |D_s|_{e^{-r}}^{-1/s})\}.$$

To improve upon this result, one might like to try to read off the generic radius of convergence, and maybe even the other subsidiary radii, from the Newton polygon of a cyclic vector. In order to do this, we have to overcome two obstructions.

(a) Some of the subsidiary radii may be greater than $p^{-1/(p-1)}\rho$, in which case Newton polygons will not detect them.

(b) One can only construct cyclic vectors in general for differential modules over differential fields, not over differential rings.

The first problem will be addressed using Frobenius descendants. The second problem will be addressed by first using a cyclic vector over a fraction field to establish linearity and integrality. We will then compare to a carefully chosen lattice to deduce convexity and monotonicity.

6 Twisted polynomials and Newton polygons

Throughout this section, let F be a complete nonarchimedean differential field. We need a way to detect truncated spectral norms of differential modules over F without writing down cyclic vectors.

Lemma 4 (Decomposition lemma)**.** *Let R be a complete subring of F. Let V be a finite differential module over F. Let $e_1, \dots, e_n$ be a basis of V via which D acts via a matrix N which has the same Newton and Hodge slopes less than $-\log |d|_F$, namely $r_1, \dots, r_i$. Then in the decomposition of V by spectral norm, the component of spectral norm s has dimension equal to the number of i such that $s = e^{-r_i}$.*

Proof. If N has no Hodge slopes less than $-\log |d|_F$, then $|D|_{\mathrm{tsp},V} \leq |D|_V \leq |d|_F$ and so we have nothing to check. We thus assume instead that the least Hodge slope of N is $r < -\log |d|_F$; it suffices to check that we can separate off a component of V accounting for that slope, by making a change of basis over $\mathfrak{o}_F$.

By the Hodge-Newton decomposition, we can find a matrix $U \in \mathrm{GL}_n(\mathfrak{o}_F)$ such that $U^{-1}NU$ splits as a block diagonal matrix, with the top left block accounting for the slope r. Put $N_1 = U^{-1}NU + d(U)U^{-1}$; since $|d(U)U^{-1}| \leq |d|_F$, using the perturbation theorem for characteristic polynomials, we see that the matrix $N_1 = U^{-1}NU + d(U)U^{-1}$ again has the same Newton and Hodge slopes less than $-\log |d|_F$, and moreover they agree with the corresponding slopes of N.

It is possible to repeat this process so as to obtain a convergent sequence of change-of-basis matrices; we will make this calculation in detail in the next unit. $\square$

In order to apply Lemma 4, we need to produce good bases of V. We can do this using twisted polynomials as follows. (We made a similar calculation in a previous unit, in the case where P had only one slope.)

Proposition 5. *Let $P = T^n + \sum_{i=0}^{n-1} P_i T^i \in F\{T\}$ be a monic twisted polynomial, and put $V = F\{T\}/F\{T\}P$. Let $r_1 \leq \cdots \leq r_n$ be the slopes of the Newton polygon of P, and suppose that there exist $\lambda_i \in F$ of norm e^{-r_i}. Let N be the matrix via which D acts on the basis*

$$\lambda_n^{-1} \cdots \lambda_{n-i+1}^{-1} T^i \qquad (i = 0, \dots, n-1).$$

Then the Hodge slopes of N are also $r_1, \dots, r_n$; in particular, the Hodge slopes less than $-\log |d|_F$ compute truncated spectral norms of constituents of V.

Proof. Put $\mu_i = \lambda_1 \cdots \lambda_{n-i-1}$ for $i = 0, \dots, n-1$; then

$$N = \begin{pmatrix} 0 & \cdots & 0 & \mu_0^{-1} P_0 \\ \lambda_{n-1} & \cdots & 0 & \mu_1^{-1} P_1 \\ & \ddots & & \vdots \\ 0 & \cdots & \lambda_1 & \mu_{n-1}^{-1} P_{n-1} \end{pmatrix}.$$

For $i = 1, \dots, n-1$, we have

$$\begin{aligned} |\mu_i^{-1} P_i| &= e^{r_i + \cdots + r_{n-i-1}} |P_i| \\ &\leq e^{r_1 + \cdots + r_{n-1-1}} e^{-r_1 - \cdots - r_{n-i}} \\ &\leq e^{-r_{n-i}} = |\lambda_{n-i}|. \end{aligned}$$

Thus by using column operations over $\mathfrak{o}_F$ (so as not to change the Hodge polygon), we can clear everything in the right column except $\mu_0^{-1} P_0$. By permuting rows and columns, we end up with a diagonal matrix with entries of norm $e^{-r_1}, \dots, e^{-r_n}$. This proves the claim. $\square$

7 Measuring small subsidiary radii

We are now almost ready to prove the components of Theorem 1 concerning large subsidiary radii. We postpone the proof of one more key lemma until we can illustrate how it will be needed.

Lemma 6. *Fix $c_0 > 1/(p-1) \log p$, and define*

$$f_i'(r) = \max\{f_i(r), c_0\}.$$

(a) (Linearity) For $i = 1, \dots, n$, the functions f_i' are continuous and piecewise affine.

(b) (Integrality) If $i = n$ or $f_i'(r_0) > \max\{f_{i+1}'(r_0), r_0 + c_0\}$, then in a neigborhood of r_0,

$$f_1'(r) + \cdots + f_i'(r) \in v(K^\times) + r\mathbb{Z}.$$

Consequently, the slopes of each f_i' belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}$.

(c) (Convexity) For $i = 1, \dots, n$, the function $f_1' + \cdots + f_i'$ is convex.

(d) (Monotonicity) Suppose that $\alpha = 0$. For $i = 1, \dots, n$, the slopes of $f_1' + \cdots + f_i'$ are nonpositive at any point r where $f_i'(r) > r + c_0$.

Proof. Put $F = \operatorname{Frac} K\langle \alpha/t, t/\beta \rangle$. Choose a cyclic vector for $M \otimes F$ to obtain an isomorphism $M \otimes F \cong F\{T\}/F\{T\}P$ for some monic twisted polynomial P over F. We may then apply Theorem 3 to deduce (a) and (b).

To deduce (c) and (d), we may work in a neighborhood of a single value r_0 of r. There is no harm in enlarging K, so we may assume $v(K^\times) = \mathbb{R}$. Then we may reduce to the case $r_0 = 0$ by replacing t by λt for some $\lambda \in K^\times$.

Apply Proposition 5 to construct $\lambda_{r,1}, \dots, \lambda_{r,n} \in K$ such that the basis of $M \otimes F_{e^{-r}}$ given by

$$\lambda_{r,n}^{-1} \cdots \lambda_{r,n-i+1}^{-1} T^i \qquad (i = 0, \dots, n-1)$$

satisfies the conclusion of the proposition. By Lemma 7 below, for any particular $c > 1$, we may construct a basis $m_1, \dots, m_n$ of M such that the supremum norm defined by this basis differs from the supremum norm defined by the chosen basis of $M \otimes F_1$ by a multiplicative factor of at most c. By continuity, for r sufficiently close to 0, the supremum norm defined by $m_1, \dots, m_n$ differs from the supremum norm defined by the chosen basis of $M \otimes F_{e^{-r}}$ by a multiplicative factor of at most c^2.

Let N be the matrix via which D acts on $m_1, \dots, m_n$. For r close to 0, by the previous paragraph, we can construct a change of basis matrix U_r between $m_1, \dots, m_n$ and the chosen basis of $M \otimes F_{e^{-r}}$, such that $|U_r|, |U_r^{-1}| \leq c^2$. For c sufficiently close to 1 (and r correspondingly close to 0), we may conclude that the Newton polygons of N and $N + d(U_r)U_r^{-1}$ coincide in slopes less than $-r$. The latter has the same Newton polygon as its conjugate $U_r^{-1} N U_r + U_r^{-1} d(U_r)$; by Lemma 4, we may conclude that for r near 0, the Newton polygon of N under $|\cdot|_r$ computes subsidiary radii less than $p^{-1/(p-1)} e^{-r}$. We may thus deduce (c) and (d) from Theorem 3. □

In the previous proof, we needed to approximate a basis of $M \otimes F_1$ with a basis of M; the following lemma allows us to do this.

Lemma 7 (Lattice lemma). *Let R be a complete K-subalgebra of F_1 (e.g., $K\langle \alpha/t, t/\beta \rangle$ with $1 \in [\alpha, \beta]$), and put $R' = R \cap \mathfrak{o}_{F_1}$. Let M be a finite free R-module of rank n, and let $|\cdot|_M$ be a norm on $M \otimes F_1$ compatible with $|\cdot|_1$. Assume that either:*

(a) $c > 1$ and the value group of K is not discrete; or

(b) $c \geq 1$, the value group of K is discrete, and the value group of M is the same as that of K.

Then there exists a norm $|\cdot|_M'$ on $M \otimes F_1$ such that $\{m \in M : |m|_M' \leq 1\}$ is a finite free R'-module of rank n, and $c^{-1}|m|_M \leq |m|_M' \leq c|m|_M$ for all $m \in M$.

Although we will only need to apply this when $|\cdot|$ is the supremum norm associated to some basis, we need the extra generality in order to carry out the induction in the proof (at least in case (a)).

Proof. We induct on n. Pick any $m_1 \in M$ belonging to a basis of M, so that $M_1 = M/Rm_1$ is also free. Using (a) or (b), we can rescale m_1 by an element of K to force $1 \leq |m_1|_M \leq c^{2/3}$.

Equip M_1 with the quotient norm

$$|x_1|_{M_1} = \inf_{x \in M: x + M_1 = x_1} \{|x|_M\};$$

this is a norm because M_1 is a closed subspace of M. Moreover, the infimum is always achieved in case (b). Apply the induction hypothesis to choose a basis $m_{2,1}, \dots, m_{n,1}$ of M_1 such that the supremum norm $|\cdot|'_{M_1}$ defined by $m_{2,1}, \dots, m_{n,1}$ satisfies $c^{-1/3}|x_1|_{M_1} \leq |x_1|'_{M_1} \leq c^{1/3}|x_1|_{M_1}$ for all $x_1 \in M_1$. For $i = 2, \dots, n$, choose $m_i \in M$ lifting $m_{i,1}$ such that $|m_i|_M \leq c^{1/3}|m_{i,1}|_{M_1} \leq c^{2/3}$.

Let $|\cdot|'_M$ be the supremum norm defined by $m_1, \dots, m_n$. For $a_1, \dots, a_n \in R'$, we have

$$|a_1 m_1 + \cdots + a_n m_n|_M \leq \max_{1 \leq i \leq n}\{|a_i||m_i|_M\} \leq c^{2/3} \leq c.$$

On the other hand, if $m \in M$ satisfies $|m|_M \leq 1$, we can uniquely write $m = a_1 m_1 + \cdots + a_n m_n$ with $a_i \in R$. By definition of the quotient norm, $|m|_{M_1} \leq 1$, so $|m|'_{M_1} \leq c^{1/3}$. In other words, $|a_2|, \dots, |a_n| \leq c^{1/3}$, so

$$|a_2 m_2 + \cdots + a_n m_n|_M \leq \max_{2 \leq i \leq n}\{|a_i||m_i|_M\} \leq c^{1/3}c^{2/3} = c.$$

Since $|m|_M \leq 1 \leq c$, we have $|a_1 m_1|_M \leq c$. Since $|m_1|_M \geq 1$, we have $|a_1| \leq c$. This proves the desired inequalities. □

8 Application of Frobenius

We now prove parts (a), (b), (c) of Theorem 1 without any lower bound restriction on the values of f_i. Again, it suffices to work in a neighborhood of $r = 0$.

We first prove an analogue of Lemma 6 in which c_0 can be replaced by any positive value. We will accomplish this using Frobenius descendants; if we tried to use Frobenius antecedents instead, we would encounter trouble in the boundary case $f_i(r) = 1/(p-1)\log p$ and in the case where $f_1(r) > 1/(p-1)\log p$ but $f_i(r) < 1/(p-1)\log p$.

Lemma 8. *Fix anonnegative integer j, and fix $c_j > p^{-j}/(p-1)\log p$. Define*

$$f'_i(r) = \max\{f_i(r), c_j\}.$$

(a) (Linearity) For $i = 1, \dots, n$, the functions f'_i are continuous and piecewise affine.

(b) (Integrality) If $i = n$ or $f_i'(r_0) > \max\{f_{i+1}'(r_0), r_0 + c_j\}$, then in a neigborhood of r_0,

$$f_1'(r) + \cdots + f_i'(r) \in v(K^\times) + r\mathbb{Z}.$$

Consequently, the slopes of each f_i' belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}$.

(c) (Convexity) For $i = 1, \dots, n$, the function $f_1' + \cdots + f_i'$ is convex.

Proof. We proceed by induction on j, the case $j = 0$ being Lemma 6. Let $R_1'(\rho^p), \dots, R_n'(\rho^p)$ be the subsidiary radii of $\varphi_* M \otimes F_\rho'$ in increasing order. (The normalization is chosen this way because the series variable in F_ρ' is t^p, which has norm ρ^p.) Put $g_i(r) = -\log R_i'(e^{-r})$. By the Frobenius descendant theorem, the list $g_1(pr), \dots, g_{pn}(pr)$ consists of

$$\bigcup_{i=1}^n \begin{cases} \{pf_i(r), pr + \frac{p}{p-1}\log p \ (p-1 \text{ times})\} & f_i(r) \leq r + 1/(p-1)\log p \\ \{\log p + (p-1)r + f_i(r) \ (p \text{ times})\} & f_i(r) \geq r + 1/(p-1)\log p. \end{cases}$$

Thus we may deduce (a) from the induction hypothesis.

To check (b) and (c), it suffices to handle cases where $i = n$ or $f_i(0) > \max\{f_{i+1}(0), c_j\}$. (We may linearly interpolate to establish convexity in the other cases.) In these cases, we have either $f_i(0) > 1/(p-1)\log p$, in which case in some neighborhood of $r = 0$ we have

$$g_1(pr) + \cdots + g_{pi}(pr) = p(f_1(r) + \cdots + f_i(r)) + pi \log p + (p-1)ipr, \tag{2}$$

or $f_{i+1}(0) < 1/(p-1)\log p$ or $i = n$, in which case in some neighborhood of $r = 0$ we have

$$g_1(pr) + \cdots + g_{pi+(p-1)(n-i)}(pr) = p(f_1(r) + \cdots + f_i(r)) + pn \log p + (p-1)npr. \tag{3}$$

Moreover, $f_i(0) > c_j$ if and only if $g_{pi}(0) > c_{j-1}$ for $c_{j-1} = pc_j$.

If $f_i(0) > 1/(p-1)\log p$, apply (2) and the induction hypothesis to write piecewise

$$\begin{aligned} f_1(r) + \cdots + f_i(r) &= p^{-1}(g_1(pr) + \cdots + g_{pi}(pr) + pi \log p + (p-1)ipr) \\ &= p^{-1}(m(pr) + *) \\ &= mr + p^{-1}* \end{aligned}$$

for some $m \in \mathbb{Z}$. (Note that $*$ is not guaranteed to be in $p \cdot v(K^\times)$; this explains the example of §4.) If $f_i(0) \leq 1/(p-1)\log p$, then $f_{i+1}(0) < 1/(p-1)\log p$, so we may apply (3) to write piecewise

$$\begin{aligned} f_1(r) + \cdots + f_i(r) &= p^{-1}(g_1(pr) + \cdots + g_{pi+(p-1)(n-i)}(pr) + pn \log p + (p-1)npr) \\ &= p^{-1}(m(pr) + *) \\ &= mr + p^{-1}* \end{aligned}$$

for some $m \in \mathbb{Z}$. (Here it was important that the domains of applicability of (2) and (3) overlap: if $f_i(0) = 1/(p-1)\log p$, then (2) may not remain applicable when we move from $r = 0$ to a nearby value.) $\square$

To finish proving (a), (b), (c) of Theorem 1, we must check them in a neighborhood of 0 under the hypothesis that $f_i(0) = 0$; note that (b) will follow immediately from (a) given that (a),(b), (c) are now known in a neighborhood of any r for which $f_i(r) > r$.

We first check continuity. By Lemma 8, for any $\epsilon > 0$, we can find $0 < \delta < \epsilon$ such that

$$|\max\{f_i(r), \epsilon/4\}| < \epsilon/2 \qquad (|r| < \delta).$$

For such r, $-\epsilon < -\delta < f_i(r) < \epsilon$; this yields continuity.

We next check piecewise affinity by induction on i. Given that $f_1, \dots, f_{i-1}$ are linear in a one-sided neighborhood of $r = 0$, say $[-\delta, 0]$, and given $f_i(0) = 0$, it suffices to check linearity of $f_i(r) - r$ in some $[-\delta', 0]$. From what we know already, in a neighborhood of each $r \in [-\delta, 0]$ where $f_i(r) - r > 0$, $f_i(r) - r$ is convex and piecewise affine with slopes in $\frac{1}{n!}\mathbb{Z}$. Note that none of these slopes can be nonnegative, otherwise $f_i(r) - r$ would thereafter be nondecreasing and could not have limit 0 at $r = 0$. By the same argument, if $f_i(r_0) - r_0 = 0$ for some $r_0 \in [-\delta, 0)$, then the slope of $f_i(r) - r$ at any point $r \in (r_0, 0)$ with $f_i(r) - r > 0$ must simultaneously be positive and negative; since this cannot occur, we must have $f_i(r) - r = 0$ for all $r \in [r_0, 0]$.

If $f_i(r) - r = 0$ for some $r < 0$, we are then done, as $f_i(r) - r$ is constant in a one-sided neighborhood of 0. Otherwise, the slopes of $f_i(r) - r$ in $[-\delta, 0)$ form a sequence of discrete values which are negative and nondecreasing. This sequence must then stabilize, so $f_i(r) - r$ is linear in a one-sided neighborhood of 0.

We finally check convexity by induction on i. Given that $f_1 + \cdots + f_{i-1}$ is convex and that $f_i(0) = 0$, it suffices to check that $f_i(r) - r$ is convex in a neighborhood of 0. But we already know that $f_i(r) - r$ is continuous and piecewise affine near 0; it must then have nonpositive left slope and nonnegative right slope, and so must be convex near 0.

9 Monotonicity

We still must prove (d) of Theorem 1. Note that we have the desired statement as part of Lemma 6 but not Lemma 8; that is because the Frobenius descendant has a pole at $t = 0$, throwing off the bound on slopes. To fix this, we must use the off-centered Frobenius descendant theorem.

Lemma 9. *If $\alpha = 0$ and $f_i(0) > 0$, then the slope in a right neighborhood of $r = 0$ is nonpositive.*

Proof. We proceed as in the proof of Lemma 8, but using the off-centered Frobenius ψ instead of φ. Let $R''_1(\rho^p), \dots, R''_n(\rho^p)$ be the subsidiary radii of $\psi_* M \otimes F''_\rho$ in increasing order. Put $g_i(r) = -\log R''_i(e^{-r})$. By the Frobenius descendant theorem, if $f_i(0) > 1/(p-1)\log p$, then

$$g_1(pr) + \cdots + g_{pi}(pr) = p(f_1(r) + \cdots + f_i(r)) + pi \log p,$$

whereas if $f_{i+1}(0) \leq 1/(p-1)\log p$ or $i = n$, then

$$g_1(pr) + \cdots + g_{pi+(p-1)(n-i)}(pr) = p(f_1(r) + \cdots + f_i(r)) + pn \log p.$$

Moreover, $f_i(0) > c_j$ if and only if $g_{pi}(0) > c_{j-1}$ for $c_{j-1} = pc_j$. Again, we deduce the claim from the corresponding claim about $\psi_* M$; this sets up an induction with base case treated by Lemma 6. $\square$

10 Subharmonicity

It is also worth noting the following harmonicity result. For $\overline{\mu} \in \kappa_K^{\mathrm{alg}}$, let μ be a lift of $\overline{\mu}$ in some complete extension L of K. If $\alpha \leq 1 \leq \beta$, let $T_\mu : K\langle \alpha/t, t/\beta \rangle \to L\langle \alpha/t, t/\beta \rangle$ be the map $t \mapsto t + \mu$.

Let $s_{\infty,i}$ be the left slope of $f_1(M,r) + \cdots + f_i(M,r)$ at $r = 0$. Let $s_{\mu,i}$ be the right slope of $f_1(T_\mu^* M, r) + \cdots + f_i(T_\mu^* M, r)$ at $r = 0$. Define the i-th *discrepancy* of M at $r = 0$ to be the sum

$$\mathrm{disc}_i(M, 0) = -\sum_{\overline{\mu} \neq 0} s_{\mu,i};$$

it is always nonnegative by Theorem 1(d). (Note that if we extend K to some larger field and consider $\overline{\mu}$ transcendental over the original residue field, then $s_{\mu,i} = 0$; that is, the definition of discrepancy is impervious to extending K.) We may extend the definition to other values of r by rescaling t.

Theorem 10 (Subharmonicity). *Assume that κ_K is algebraically closed and that $1 \in (\alpha, \beta)$. Fix $i \in \{1, \dots, n\}$ such that $f_i(0) > 0$. Then*

$$s_{0,i} - s_{\infty,i} \geq \mathrm{disc}_i(M, 0),$$

with equality if $i = n$.

Proof. By applying φ_* as needed, we can force $f_i(r) > 1/(p-1) \log p$. Then this follows from an argument analogous to Lemma 6, but with Lemma 11 below used in place of Theorem 3. $\square$

Lemma 11. *Assume that κ_K is algebraically closed and that $1 \in (\alpha, \beta)$. For $f \in K\langle \alpha/t, t/\beta \rangle$ nonzero, put $v_r(f) = -\log |f|_{e^{-r}}$ for $r \in [-\log \beta, -\log \alpha]$. Let s_∞ be the left slope of $v_r(f)$ at $r = 0$. Let s_μ be the right slope of $v_r(T_\mu(f))$ at $r = 0$. Then*

$$s_\infty = \sum_\mu s_\mu.$$

Proof. Without loss of generality, we may assume that $|f|_1 = 1$. The quotient of $\mathfrak{o}_{F_1} \cap K\langle \alpha/t, t/\beta \rangle$ by the ideal generated by $\mathfrak{m}_F$ is isomorphic to $\kappa_K[t, t^{-1}]$; let $\overline{f}$ be the image of f in this quotient. Then s_μ is the order of vanishing of f at μ, whereas s_∞ is the negative of the order of vanishing of f at ∞. This gives the desired equality. $\square$

Since discrepancy is nonnegative, Theorem 10 includes the convexity inequality $s_{\infty,i} \leq s_{0,i}$ from Theorem 1(c). One can turn things around to get the following corollary.

Corollary 12. *With notation as in Theorem 10, if $s_{\infty,i} = s_{0,i}$, then $s_{\mu,i} = 0$ for all $\overline{\mu} \neq 0$.*

We also have a corollary that says that for a finite differential module over $K\langle \alpha/t, t/\beta \rangle$, the generic radius of convergence can be computed at any point in all but finitely many residue discs, not just in a generic residue disc.

Corollary 13. *With notation as in Theorem 10, $s_{\mu,i} = 0$ for all but finitely many $\overline{\mu}$.*

Proof. The $s_{\mu,i}$ lie in the discrete subgroup $\frac{1}{n!}\mathbb{Z}$ of $\mathbb{R}$ and are nonpositive, so their sum can only be bounded below if all but finitely many of them are zero. $\square$

11 Radius and generic radius

We can now interpret the radius of convergence of a differential module on a disc in terms of the function f_1.

Proposition 14. *Let M be a differential module over $K\langle t/\beta \rangle$ for some $\beta > 0$. Then the radius of convergence of M equals e^{-r}, for r the smallest value such that $f_1(r) = r$. Consequently, $f(r') = r'$ for all $r' \geq r$.*

Proof. By a result from a previous unit, the radius of convergence of M is at least the generic radius of convergence of $M \otimes F_{e^{-r}}$, which by hypothesis equals e^{-r}. On the other hand, if $\lambda > e^{-r}$, then by hypothesis $f_1(-\log \lambda) > -\log \lambda$, or in other words $R(M \otimes F_\lambda) < \lambda$. This means that $M \otimes K\langle t/\lambda \rangle$ cannot be trivial, so the radius of convergence cannot exceed λ. This proves the desired result. $\square$

Corollary 15. *Let M be a differential module over $K\langle t/\beta \rangle$ for some $\beta > 0$. Then the radius of convergence of M belongs to the divisible closure of the multiplicative value group of K.*

Proof. By Theorem 1 and Theorem 2, the function $f_1(r)$ is piecewise of the form $ar + b$ with $a \in \mathbb{Q}$ and b in the divisible closure of the additive value group of K. By Proposition 14, the radius of convergence of M equals e^{-r} for r the smallest value such that $f_1(r) = r$. To the left of this r, f_1 must be piecewise affine with slope $\neq 1$; by comparing the left and right limits at r, we deduce that $r = ar + b$ for some $a \neq 1$ rational and some b in the divisible closure of the additive value group of K. Since this gives $r = b/(a-1)$, we deduce the claim. $\square$

One should be able to better control the denominators, as in the following question.

Question 16. *Let M be a differential module over $K\langle t/\beta \rangle$ for some $\beta > 0$. Does there necessarily exist $j \in \{1, \dots, \operatorname{rank}(M)\}$ such that the j-th power of the radius of convergence of M belongs to the p-divisible closure of the multiplicative value group of K?*

We also have a criterion for when the radius of convergence equals the generic radius.

Corollary 17. *Let M be a differential module over $K\langle t/\beta \rangle$ for some $\beta > 0$, such that for some $\alpha \in (0, \beta)$, $R(M \otimes F_\rho)$ is constant for $\rho \in [\alpha, \beta]$. Then $R(M) = R(M \otimes F_\rho)$. (A similar statement holds for the product of the first i subsidiary radii, for $i = 1, \dots, \operatorname{rank}(M)$.)*

12 Subsidiary radii as radii of convergence

The generic radii of subsidiary convergence can be interpreted as the radii of convergence of a well-chosen basis of local horizontal sections at a generic point.

Theorem 18 (after Young)**.** *Let (V, D) be a differential module over F_ρ of dimension n with subsidiary radii $s_1 \leq \cdots \leq s_n$. Choose a basis $e_1, \ldots, e_n$ of local horizontal sections of V at a generic point η. For $i = 1, \ldots, n$, let ρ_i be the radius of convergence of e_i, and suppose that $\rho_1 \leq \cdots \leq \rho_n$. Then $\rho_i \leq s_i$ for $i = 1, \ldots, n$; moreover, there exists a choice of basis for which $\rho_i = s_i$ for $i = 1, \ldots, n$.*

Proof. We first produce a basis for which $\rho_i = s_i$ for $i = 1, \ldots, n$. For this, we may apply the strong decomposition theorem to decompose V into components each with a single subsidiary radius, and thus reduce to the case $s_1 = \cdots = s_n = s$. By the geometric interpretation of generic radius, each Jordan-Hölder constituent of V admits a basis of local horizontal sections on a generic disc of radius s. By a prior lemma, the same is true for V itself.

For the remaining inequality, we induct on n. Let m be the largest integer such that $s_1 = s_m$. Let V_1 be the component of V of subsidiary radius s_1, so that $\dim V_1 = m$. We will check that no local horizontal section of V_1 at a generic point η can have radius of convergence strictly greater than s_1.

Suppose the contrary; then there would exist a local horizontal section of V_1 at η which converges on a closed disc of radius λ for some $\lambda \in (s_1, \rho)$. This would mean that $V_1 \otimes L\langle (t-\eta)/\lambda \rangle$ would have a trivial submodule, and so would have λ as one of its subsidiary radii. However, by arguing as in Theorem 10, we see that the product of the subsidiary radii of $V_1 \otimes L\langle (t-\eta)/\lambda \rangle$ is equal to s_1^m for λ slightly smaller than ρ; by Theorem 1(c) and (d), the equality holds for all $\lambda \in (s_1, \rho)$. This yields a contradiction.

We conclude that any local horizontal section of V that projects nontrivially onto V_1 has radius strictly greater than s_1. We can divide the given basis into m sections that project onto a basis of V_1, and $n - m$ sections that project onto a basis of the complementary component. The first m sections have radius of convergence at most s_1 by above; the others have radii of convergence bounded by $s_{m+1}, \ldots, s_n$ by the induction hypothesis. This yields the desired result. $\square$

A basis of local horizontal sections for which $\rho_i = s_i$ for $i = 1, \ldots, n$ is sometimes called an *optimal basis.*

13 Notes

For f_1, Christol and Dwork established convexity [CD04, Proposition 2.4] (using essentially the same short proof given here) and continuity at endpoints [CD04, Théorème 2.5]. All other results in this unit are original.

We again remind the reader that subharmonicity, as a property of suitable functions on Berkovich spaces, is addressed in the work of Thuillier [Thu05].

When restricted to intrinsic subsidiary radii less than $p^{-1/(p-1)}$, Theorem 18 is a result of Young [You92, Theorem 3.1]. Young's proof is an explicit calculation using twisted polynomials; it was limited to small radii because the Frobenius antecedent theorem was not available at the time.

14 Exercises

1. Given an example to show that in Theorem 3, f_2 need not be concave (even though f_1 and f_2 are concave).

2. Here is a result of Dwork related to the example in § 4. Suppose $\pi \in K$ satisfies $\pi^{p-1} = -p$. Prove that the power series $E(t) = \exp(\pi t - \pi t^p)$ has radius of convergence strictly greater than 1. (By contrast, the series $\exp(\pi t)$ has radius of convergence 1.) Optional: prove that the radius of convergence is equal to $p^{(p-1)/p^2}$.

3. Prove that if K is discretely valued, then $\mathfrak{o}_K\langle t\rangle$ is noetherian. It isn't otherwise, because then $\mathfrak{o}_K$ itself is not noetherian.

4. Prove that each maximal ideal of $\mathfrak{o}_K\langle t\rangle$ is generated by $\mathfrak{m}_K$ together with some $P \in \mathfrak{o}_K[t]$ whose reduction modulo $\mathfrak{m}_K$ is irreducible in $\kappa_K[t]$.