p-adic differential equations
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Decomposition by subsidiary radii

In this unit, we show that one can sometimes decompose a differential module on a disc or annulus so as to separate the small and large subsidiary radii into different components.

1 Unit coefficients in Newton polygons

For any complete extension L of K and any $\mu \in \mathfrak{o}_L$, we write $f_\mu : K\langle t \rangle \to L\langle t \rangle$ for the map $t \mapsto t + \mu$.

Lemma 1. *For $x \in K\langle t \rangle$, x is a unit if and only if for any L and μ, $|f_\mu(x)|_{e^{-r}}$ is independent of r. (By concavity, it suffices to check this for r in a neighborhood of 0.)*

Proof. Write $x = \sum_i x_i t^i$. Then x is a unit if and only if $|x_0| > |x_i|$ for all $i > 0$. If this holds, then clearly $|f_\mu(x)|_{e^{-r}} = |x_0|$ for all μ, r. Otherwise, we can reduce $x_0^{-1}x$ modulo $\mathfrak{m}_K$ to get a nonconstant element of $\kappa_K[t]$; by making L large enough, we can choose μ to reduce to a root of this polynomial, and then $|f_\mu(x)|_{e^{-r}}$ will not be constant. □

This immediately yields the following.

Theorem 2. *Let $P = T^n + \sum_{i=0}^{n-1} P_i T^i \in K\langle t \rangle\{T\}$ be a monic twisted polynomial. Choose $j \in \{1, \dots, n-1\}$ such that there is a vertex of the Newton polygon of P, measured with respect to $|\cdot|_1$, with x-coordinate $(-n + j)$, and that the first j slopes are all less than $-\log |d|_{F_1}$. Then P_j is a unit in $K\langle t \rangle$ if and only if for any L and μ, the sum of the first j slopes of the Newton polygon of $f_\mu(P) = T^n + \sum_{i=0}^{n-1} f_\mu(P_i)T^i$, measured with respect to $|\cdot|_{e^{-r}}$, is independent of r. (By concavity, it suffices to check this for r in a neighborhood of 0.) Moreover, in this case (by the master factorization theorem), there is a factorization of P separating the first j slopes.*

For annuli, it is a bit more complicated to formulate the analogous results; here is a weak but adequate formulation.

Lemma 3. *For $x \in \cup_{\alpha \in (0,1)} K\langle \alpha/t, t \rangle$, x is a unit if and only if for any L and μ, $|f_\mu(x)|_{e^{-r}}$ is independent of r for r in a neighborhood of 0.)*

Theorem 4. *Let $P = T^n + \sum_{i=0}^{n-1} P_i T^i \in (\cup_{\alpha \in (0,1)} K\langle \alpha/t, t \rangle)\{T\}$ be a monic twisted polynomial. Choose $j \in \{1, \dots, n-1\}$ such that there is a vertex of the Newton polygon of P, measured with respect to $|\cdot|_1$, with x-coordinate $(-n + j)$, and that the first j slopes are all less than $-\log |d|_{F_1}$. Then P_j is a unit in $\cup_{\alpha \in (0,1)} K\langle \alpha/t, t \rangle$ if and only if for any L and μ, the sum of the first j slopes of the Newton polygon of $f_\mu(P) = T^n + \sum_{i=0}^{n-1} f_\mu(P_i)T^i$, measured with respect to $|\cdot|_{e^{-r}}$, is independent of r for r in a neighborhood of 0. Moreover, in this case (by the master factorization theorem), there is a factorization of P separating the first j slopes.*

2 Decomposition over a disc

Retain notation as in the subharmonicity theorem from the previous unit.

Theorem 5. *Let M be a finite differential module over $K\langle t/\beta\rangle$ of rank n. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) We have $f_i(-\log\beta) > f_{i+1}(-\log\beta)$.

(b) The function $f_1 + \cdots + f_i$ is constant in a neighborhood of $r = -\log\beta$.

(c) The i-th discrepancy of M is zero at $r = -\log\beta$.

Then the decomposition of $M \otimes F_\beta$ separating the first i subsidiary radii lifts to a decomposition of M itself.

Before proving Theorem 5, we record a trivial but useful observation.

Lemma 6. *Let R, S, T be subrings of a common ring U with $S \cap T = R$. Let M be a finite free R-module. Then the intersection $(M \otimes S) \cap (M \otimes T)$ inside $M \otimes U$ is equal to M itself.*

This also holds when M is only locally free; see exercises. The immediate application is to replace K by a complete extension L in Theorem 5; inside the completion of $L(t)$ for the 1-Gauss norm, we have

$$F_1 \cap L\langle t\rangle = K\langle t\rangle.$$

Thus obtaining matching decompositions of $M \otimes F_1$ and $M \otimes L\langle t\rangle$ gives a corresponding decomposition of M itself.

Lemma 7. *Theorem 5 holds if $f_i(-\log\beta) > 1/(p-1)\log p - \log\beta$.*

Proof. As in the previous unit, we may reduce to the case $\beta = 1$. (This requires enlarging K, but as noted above, Lemma 6 renders this enlargement harmless.) Set notation as in Lemma 6 from the previous unit. Then for r near 0, the sum of the first i slopes of characteristic polynomial of N equals the sum of the first i slopes of the Newton polygon of the twisted polynomial P, which is constant, and the i-th slope on either side is strictly less than the $(i+1)$-st slope. Moreover, this all remains true after replacing t by $t + c$ for any c of norm 1. Consequently, by Theorem 2, the coefficient of T^{n-i} in the characteristic polynomial of N is a unit in $K\langle t\rangle$, and serves as a vertex of the Newton polygon for all r. We may thus perform a slope decomposition in $K\langle t\rangle[T]$ to separate the first i slopes.

By applying the lattice lemma from the previous unit (again with K suitably large) to both factors of the decomposition, we obtain a matrix $U_1 \in \mathrm{GL}_n(K\langle t\rangle)$ with $|U_1|, |U_1|^{-1} \le c$ such that $U_1^{-1} N U_1$ is a block diagonal matrix, with the first i Newton slopes in the first block and the rest in the second block.

Put $N_1 = U_1^{-1} N_0 U_1 + U_1^{-1} d(U_1)$. Let $s_{H,1}, \dots, s_{H,n}$ be the Hodge slopes of N_1; they differ from the corresponding Newton slopes by at most $4\log c$. We now repeat the following operation to produce $N_2, N_3, \dots$. Suppose we have

$$N_l = \begin{pmatrix} A_l & B_l \\ C_l & D_l \end{pmatrix}$$

with A_l having Hodge slopes $s_{H,1}, \dots, s_{H,i}$, and $|B_l|, |C_l|, |D_l| \leq e^{-s_{H,i}}\delta$ for some $\delta < 1$. (These hold for $l = 1$ if we take c small enough.) Put

$$U_{l+1} = \begin{pmatrix} I_i & 0 \\ C_l A_l^{-1} & I_{n-i} \end{pmatrix}.$$

Note that $|A_l^{-1}| \leq e^{s_{H,i}}$ by Cramer's rule, so $|U_{l+1} - I_n| \leq \delta < 1$. In particular, U_{l+1} is invertible. Set

$$N_{l+1} = U_{l+1}^{-1} N_l U_{l+1} + U_{l+1}^{-1} d(U_{l+1}).$$

In block form,

$$N_{l+1} = \begin{pmatrix} A_l - B_l C_l A_l^{-1} & B_l \\ -C_l A_l^{-1} B_l C_l A_l^{-1} + D_l C_l A_l^{-1} + d(C_l A_l^{-1}) & -C_l A_l^{-1} B_l + D_l \end{pmatrix},$$

so

$$\begin{aligned} \max\{|B_{l+1}|, |D_{l+1}|\} &\leq e^{-s_{H,i}}\delta \\ |C_{l+1}| &\leq \max\{e^{s_{H,i}}, \delta\}|C_l|. \end{aligned}$$

Since $\max\{e^{s_{H,i}}, \delta\} < 1$, $C_l \to 0$ and so $U_l \to I_n$.

In the limit, we get a basis of M on which D acts by a block upper triangular matrix; this shows that the first component of the decomposition of $M \otimes F_1$ separating the first i subsidiary radii from the others lifts to M. Repeating the argument with M replaced by $M^\vee$ shows that the second component also lifts (because its annihilator in $M^\vee \otimes F_1$ lifts). ☐

To prove Theorem 5 in general, we must reduce the problem for M to the problem for $\varphi_* M$. If $M_1' \oplus M_2'$ is the corresponding decomposition of $\varphi_* M$, then this might not be induced by a decomposition of M_1, because some factors of subsidiary radius $p^{-p/(p-1)}$ that are needed in M_2' are instead grouped into M_1'. To fix this, consider instead the decomposition

$$((M_1' \otimes W_0) \cap \cdots \cap (M_1' \otimes W_{p-1})) \oplus ((M_2' \otimes W_0) + \cdots + (M_2' \otimes W_{p-1}));$$

this is induced by the desired decomposition of M_1.

3 Decomposition over a closed annulus

A similar argument works for annuli, to prove the following.

Theorem 8. *Let M be a finite differential module over $K\langle \alpha/t, t/\beta \rangle$ of rank n. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) We have $f_i(r) > f_{i+1}(r)$ for $-\log\beta \leq r \leq -\log\alpha$.

(b) The function $f_1 + \cdots + f_i$ is affine on $[-\log\beta, -\log\alpha]$.

(c) The i-th discrepancy of M is zero at both $r = -\log\beta$ and $r = -\log\alpha$.

Then there is a decomposition of M inducing, for each $\rho \in [\alpha, \beta]$, the decomposition of $M \otimes F_\rho$ separating the first i subsidiary radii from the others.

In the following proof, we will skip some details which proceed as in the disc case.

Proof. By the patching theorem for finite free modules over $K\langle\alpha/t, t/\beta\rangle$, it suffices to show that we can cover the interval $[\alpha, \beta]$ with finitely many closed subintervals $[\gamma, \delta]$ of positive length, such that the desired conclusion holds for $M \otimes K\langle\gamma/t, t/\delta\rangle$. Since the interval $[\alpha, \beta]$ is compact, it suffices to check that each $\rho \in [\alpha, \beta]$ occurs as the endpoint of such a good interval.

Note that condition (b) implies that the i-th discrepancy of M is zero for all $r \in (-\log\beta, -\log\alpha)$. That means that we may relabel things so that ρ becomes an endpoint of the interval, without losing our hypotheses. That is, it suffices to check that there exists $\gamma \in [\alpha, \beta)$ such that the desired decomposition exists for $M \otimes \langle\gamma/t, t/\beta\rangle$.

Using Lemma 6, we may enlarge K and then reduce to the case $\beta = 1$. Moreover, using Frobenius descendants as in the proof of Theorem 5, we may reduce to the case where $f_i(0) > 1/(p-1)\log p$.

Again, set notation as in Lemma 6 from the previous unit. This time, hypotheses (a) and (c) plus Theorem 4 imply that for $\gamma \in [\alpha, 1)$ sufficiently close to 1, the coefficient of T^{n-i} in the characteristic polynomial of N is a unit in $K\langle\gamma/t, t\rangle$, and serves as a vertex of the Newton polygon for all r. We may thus perform a slope decomposition in $K\langle\gamma/t, t\rangle[T]$ to separate the first i slopes, and continue as in Lemma 7. □

4 Decomposition over an open disc or annulus

If we are willing to lose information at the endpoints of our given interval, we can drop condition (c) from Theorem 8.

Theorem 9. *Let M be a finite differential module over $K\langle t/\beta\rangle$ of rank n. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) For r in a neighborhood of $-\log\beta$, $f_1 + \cdots + f_i$ is constant.

(b) For $r \in (-\log\beta, \infty)$ sufficiently small, $f_i(r) > f_{i+1}(r)$. (We do not require the inequality for $r = -\log\beta$.)

Then for any $\delta < \beta$, $M \otimes K\langle t/\delta\rangle$ admits a unique decomposition separating the first i subsidiary radii.

Proof. It suffices to check this for a sequence of values of δ approaching β. To do this, note that by subharmonicity, the ith discrepancy of M can only be nonzero at a value of r where the function $f_1 + \cdots + f_i$ has a change of slope. In particular, those values occur discretely in any interval, so for $\delta < \beta$ sufficiently close to β, $-\log\delta$ is not such a value. We may then apply Theorem 5 to conclude. □

We have a similar result for annuli; the proof is similar, so we omit it.

Theorem 10. *Let M be a finite differential module over $K\langle\alpha/t, t/\beta\rangle$ of rank n. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) We have $f_i(r) > f_{i+1}(r)$ for $-\log\beta < r < -\log\alpha$. (We do not require the inequality for $r = -\log\beta$ or $r = -\log\alpha$.)

(b) The function $f_1 + \cdots + f_i$ is affine on $[-\log\beta, -\log\alpha]$.

Then for any $\alpha < \gamma \leq \delta < \beta$, $M \otimes K\langle\gamma/t, t/\delta\rangle$ admits a unique decomposition separating the first i subsidiary radii.

5 Modules solvable at a boundary

Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius β. We say M is *solvable at β* if $R(M \otimes F_\rho) \to \beta$ as $\rho \to \beta^-$, or equivalently, if $IR(M \otimes F_\rho) \to 1$ as $\rho \to \beta^-$.

Lemma 11. *Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius β, which is solvable at β. There exist $b_1 \geq \cdots \geq b_n \in [0, \infty)$ such that for $\rho \in [\alpha, \beta)$ sufficiently close to β, the intrinsic subsidiary radii of $M \otimes F_\rho$ are $(\rho/\beta)^{b_1}, \dots, (\rho/\beta)^{b_n}$. Moreover, if $i = n$ or $b_i > b_{i+1}$, then $b_1 + \cdots + b_i \in \mathbb{Z}$.*

Proof. Define $f_1(r), \dots, f_n(r)$ as before. For $r \to (-\log\beta)^+$, $f_1 + \cdots + f_i - ir$ is convex with slopes in a discrete subset of $\mathbb{R}$ and has limit 0; this implies that the slopes are all positive. However, the slopes lie in a discrete subgroup of $\mathbb{R}$, so they must eventually stabilize. We deduce that each f_i is linear in a neighborhood of $-\log\beta$, and we may infer the desired conclusions from the known properties of the f_i. □

The quantities $b_1, \dots, b_n$ are called the *(differential) slopes* of M at β. We will have a closer look at these numbers in a later unit.

We now recover a decomposition theorem of Christol-Mebkhout; see notes.

Theorem 12 (Christol-Mebkhout)**.** *Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius β, which is solvable at β. Then for some $\gamma \in [\alpha, \beta)$, the restriction of M to the annulus with closed inner radius α and open outer radius β splits as a direct sum $\oplus_{b \in [0,\infty)} M_b$, such that for each $b \in [0, \infty)$, for all $\rho \in [\gamma, \beta)$, the intrinsic subsidiary radii of $M_b \otimes F_\rho$ are all equal to $(\rho/\beta)^b$.*

Proof. By Lemma 11, we are in a case where Theorem 10 may be applied. □

The condition of solvability at a boundary is often established by exhibiting a Frobenius structure for the differential equation. We will explain what this means in a later unit.

6 Notes

Our results on modules solvable at a boundary are originally due to Christol and Mebkhout [CM00], [CM01]. In particular, Lemma 11 for the generic radius is [CM00, Théorème 4.2.1], and the decomposition theorem (which implies Lemma 11 in general) is [CM01, Corollaire 2.4–1]. The proof technique of Christol and Mebkhout is significantly different from ours: they construct the decomposition by exhibiting convergent sequences for a certain topology on the ring of differential operators.

Note that Christol and Mebkhout do not obtain quantitative results; that is, they do not control the range over which the decomposition occurs. We are not sure whether this is really a limitation of their method, or whether they did not bother to exert such control because they did not have an application in mind for it. (Remember that for the quantitative results here, it was crucial to use Frobenius descendants, which do not appear previously elsewhere.)

Note also that Christol and Mebkhout work directly with a differential module on an open annulus as a ring-theoretic object; this requires a freeness result of the following form. If K is spherically complete, any finite free module on the half-open annulus with closed inner radius α and open outer radius β is induced by a finite free module over the ring $\cap_{\rho \in [\alpha,\beta)} K\langle \alpha/t, t/\rho \rangle$. (That is, any locally free coherent sheaf on this annulus is freely generated by global sections.) For a proof, see for instance [Ked05, Theorem 3.14]. A result of Lazard [Laz62] implies that this property, even when restricted to modules of rank 1, is in fact equivalent to spherical completeness of K.

7 Exercises

1. Prove the analogue of Lemma 6 where M is only locally free. This is [Ked07b, Lemma 1.2.7].