p-adic differential equations
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A little difference algebra

In this unit, we set up a bit of formalism for difference algebra, parallel to what we did with differential algebra earlier.

1 Difference algebra

A *difference ring/field* is a ring/field R equipped with an endomorphism ϕ. A *difference module* over R is an R-module M equipped with a map $\Phi : R \to R$ which is additive and ϕ-semilinear; the latter means that

$$\Phi(rm) = \phi(r)\Phi(m) \qquad (r \in R, m \in M).$$

A difference submodule of R itself is also called a *difference ideal*.

If M is a finite difference module over R freely generated by $e_1, \dots, e_n$, then we can recover the action of Φ from the $n \times n$ matrix A defined by

$$\Phi(e_j) = \sum_i A_{ij} e_i.$$

Namely, if we use the basis to identify M with the space of column vectors of length n over R, then

$$\Phi(v) = A\phi(v).$$

Moreover, if we change to a new basis $e'_1, \dots, e'_n$, and let U be the change-of-basis matrix (defined by $e'_j = \sum_i U_{ij} e_i$), then Φ acts on the new basis via the matrix

$$A' = U^{-1} A \phi(U).$$

We say M is *dualizable* if A is invertible. If M is dualizable, we define the dual $M^\vee$ as the module-theoretic dual $\mathrm{Hom}_R(M, R)$ with Φ-action given on the dual basis by A^{-T} (the inverse transpose). Note that the property of dualizability, and the definition of the dual, do not depend on the choice of the basis; hence they both extend to the case where M is only locally free as an R-module.

We say that the difference ring R is *inversive* if ϕ is an automorphism. In this case, we can define the *opposite difference ring* R^{opp} to be R again, but now equipped with the endomorphism ϕ^{-1}. If R is inversive and M is locally free, we define the *opposite module* M^{opp} of M as the module-theoretic dual $\mathrm{Hom}_R(M, R)$ equipped with the pullback action (i.e., on the dual basis, use the matrix A^T for the action).

For M a difference module, write

$$H^0(M) = \ker(\mathrm{id} - \Phi), \qquad H^1(M) = \mathrm{coker}(\mathrm{id} - \Phi).$$

If M_1, M_2 are difference modules with M_1 dualizable, then $H^0(M_1^\vee \otimes M_2)$ computes morphisms from M_1 to M_2, and $H^1(M_1^\vee \otimes M_2)$ computes extensions $0 \to M_2 \to M \to M_1 \to 0$. That is,

$$H^0(M_1^\vee \otimes M_2) = \mathrm{Hom}(M_1, M_2), \qquad H^1(M_1^\vee \otimes M_2) = \mathrm{Ext}(M_1, M_2).$$

2 Twisted polynomials

As in differential algebra, there is a relevant notion of twisted polynomials. For R a difference ring, we define the twisted polynomial ring $R\{T\}$ as the set of finite formal sums $\sum_{i=0}^{\infty} r_i T^i$, but with the multiplication this time obeying the rule $Tr = \phi(r)T$. For any $P \in R\{T\}$, the quotient $R\{T\}/R\{T\}P$ is a difference module; if M is a difference module, we say $m \in M$ is a *cyclic vector* if there is an isomorphism $M \cong R\{T\}/R\{T\}P$ carrying m to 1.

If R is inversive, we again have a formal adjoint construction: given $P \in R\{T\}$, its formal adjoint is obtained by pushing the coefficients to the right side of T. This may then be viewed as an element of the opposite ring of $R\{T\}$, which we may identify with $R^{\mathrm{opp}}\{T\}$.

It is not completely straightforward to analogize the cyclic vector theorem to difference modules; see the exercises for one attempt to do so. Instead, we will use only the following trivial observation.

Lemma 1. *Any irreducible finite difference module over a difference field contains a cyclic vector.*

Proof. If F is a difference field, M is a finite difference module over F, and $m \in M$ is nonzero, then $m, \Phi(m), \dots$ generate a nonzero difference submodule of M. If M is irreducible, this submodule must be all of M. $\square$

If ϕ is isometric to a norm $|\cdot|$ on F, then we have the usual definition of Newton polygons and slopes for twisted polynomials. If R is inversive, then a twisted polynomial and its adjoint have the same Newton polygon.

Applying the master factorization theorem yields the following.

Theorem 2. *Let F be a difference field complete for a norm $|\cdot|$ under which F is isometric. Then any monic twisted polynomial $P \in F\{T\}$ admits a unique factorization*

$$P = P_{r_1} \cdots P_{r_m}$$

for some $r_1 < \cdots < r_m$, where each P_{r_i} is monic with all slopes equal to r_i. (If F is inversive, the same holds with the factors in the opposite order.)

3 Difference-closed fields

We will say that a difference field F is *weakly difference-closed* if every dualizable finite difference module over F is trivial. We say F is *strongly difference-closed* if F is inversive and weakly difference-closed.

Lemma 3. *The difference field F is weakly difference-closed if and only if the following conditions hold.*

(a) Every nonconstant monic twisted polynomial $P \in F\{T\}$ factors as a product of linear factors.

(b) For every $c \in F^\times$, there exists $x \in F^\times$ with $\phi(x) = cx$.

(c) For every $c \in F^\times$, there exists $x \in F$ with $\phi(x) - x = c$.

Proof. We first suppose that F is weakly difference-closed. To prove (a), it suffices to check that if $P \in F\{T\}$ is nonconstant monic with nonzero constant term, then P factors as $P_1 P_2$ with P_2 linear. The nonzero constant term implies that $M = F\{T\}/F\{T\}P$ is a dualizable finite difference module over F, so must be trivial by the hypothesis that F be weakly difference-closed. In particular, there exists a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ with M_2 trivial; this corresponds to a factorization $P = P_1 P_2$ with P_2 linear.

To prove (b), note that $F\{T\}/F\{T\}(T - c^{-1})$ must be trivial, which means there exists $x \in F^\times$ such that $Tx - x = y(T - c^{-1})$ for some $y \in F$. Then $y = \phi(x)$ and $yc^{-1} = x$, proving the claim.

To prove (c), form the ϕ-module V corresponding to the matrix $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$. By construction, we have a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with V_1, V_2 trivial; since V must also be trivial, this extension must split. That means that we can find $x \in F$ with $\phi(x) - x = c$, proving the claim.

Conversely, suppose that (a), (b), (c) hold. Every nonzero dualizable finite difference module over F admits an irreducible quotient. This quotient admits a cyclic vector by Lemma 1, and so admits a quotient of dimnesion 1 by (a). That quotient in turn is trivial by (b). By induction, we deduce that every dualizable finite difference module over F admits a filtration whose successive quotients are trivial of dimension 1. This filtration splits by (c). □

Proposition 4. *Let F be a separably (resp. algebraically) closed field of characteristic $p > 0$ equipped with a power of the absolute Frobenius. Then F is weakly (resp. strongly) difference-closed.*

Proof. For $P = \sum_{i=0}^m P_i T^m \in F\{T\}$ with $m > 0$, $P_m = 1$, and $P_0 \neq 0$, the polynomial $Q(x) = \sum_{i=0}^m P_i x^{q^i}$ has degree $q^m \geq 2$, and $x = 0$ occurs as a root only with multiplicity 1. Moreover, the formal derivative of P is a constant polynomial, so has no common roots with P; hence P is a separable polynomial. Since F is separably closed, there must exist a nonzero root x of Q; this implies criteria (a) and (b) of Lemma 3. To deduce (c), note that for $c \in F^\times$, the polynomial $x^q - x - c$ is again separable, so has a root in F. □

4 Difference algebra over a complete field

For the rest of this unit, let F be a difference field complete for a norm $|\cdot|$ with respect to which ϕ is isometric. We do not assume that F is inversive; if not, then we can embed into

F into an inversive difference field by forming the completion F' of the direct limit of the system
$$F \xrightarrow{\phi} F \xrightarrow{\phi} \cdots .$$

As in the differential case, we would like to classify finite difference modules over F by the spectral norm of Φ (there is no truncation here); the following basic properties will help, as long as we are mindful of the discrepancies between the differential and difference cases.

Lemma 5. *Let V, V_1, V_2 be finite difference modules over F.*

(a) For $0 \to V_1 \to V \to V_2 \to 0$ a short exact sequence,
$$|\Phi|_{\mathrm{sp},V} = \max\{|\Phi|_{\mathrm{sp},V_1}, |\Phi|_{\mathrm{sp},V_2}\}.$$

(b) We have
$$|\Phi|_{\mathrm{sp},V_1 \otimes V_2} = |\Phi|_{\mathrm{sp},V_1} |\Phi|_{\mathrm{sp},V_2}.$$

(c) We have
$$|\Phi|_{\mathrm{sp},V} = |\Phi|_{\mathrm{sp},V \otimes F'}.$$

Proof. Exercise. □

The relationship between V and the dual $V^\vee$ is more complicated.

Lemma 6. *If $V \cong F\{T\}/F\{T\}P$ and P has only one slope r in its Newton polygon, then*
$$|\Phi|_{\mathrm{sp},V} = e^{-r}.$$
If F is inversive, then also
$$|\Phi^{-1}|_{\mathrm{sp},V} = e^{-r}.$$

Proof. By replacing F with F', we may reduce to the case where F is inversive. Pick a positive integer d such that there exists $\lambda \in F$ such that $|\lambda| = e^{-rd}$. Then the basis of $F\{T\}/F\{T\}P$ given by $(\lambda^{-1}\Phi^d)^i(1)$ for $i = 0, \dots, (\dim_F V) - 1$ has the property that
$$|\Phi^d|_V = e^{-rd}.$$
We deduce that
$$|\Phi|_{\mathrm{sp},V} \leq e^{-r};$$
since
$$|\Phi^{-1}|_{\mathrm{sp},V} = |\Phi|_{\mathrm{sp},V^{\mathrm{opp}}},$$
and $V^{\mathrm{opp}} \cong F^{\mathrm{opp}}\{T\}/F\{T\}Q$ for Q the formal adjoint of P, we also have
$$|\Phi^{-1}|_{\mathrm{sp},V} \leq e^{r}.$$
Since
$$1 = |\Phi|_{\mathrm{sp},V} |\Phi^{-1}|_{\mathrm{sp},V} \leq e^{-r} e^{r},$$
we obtain the desired equalities. □

Corollary 7. *For any finite difference module V over F, either $|\Phi|_{\mathrm{sp},V} = 0$, or there exists an integer $m \in \{1, \dots, \dim_F V\}$ such that $|\Phi|^m_{\mathrm{sp},V} \in |F^\times|$.*

Let V be a nonzero finite difference module over F. We say that V is *pure of norm s* if all of the Jordan-Hölder constituents of V have spectral norm s. Note that V is pure of norm 0 if and only if $\Phi^{\dim_F V} = 0$.

Proposition 8. *Let V be a finite difference module over F. Then V is pure of norm $s > 0$ if and only if*

$$|\Phi|_{\mathrm{sp},V\otimes F'} = s, \qquad |\Phi^{-1}|_{\mathrm{sp},V\otimes F'} = s^{-1}. \tag{1}$$

Proof. If V is pure of norm s, then (1) holds by Lemma 6. Conversely, if (1) holds and W is a subquotient of V, then

$$|\Phi|_{\mathrm{sp},W\otimes F'} \leq |\Phi|_{\mathrm{sp},V\otimes F'}, \qquad |\Phi^{-1}|_{\mathrm{sp},W\otimes F'} \leq |\Phi^{-1}|_{\mathrm{sp},V\otimes F'}.$$

We thus have

$$1 \leq |\Phi|_{\mathrm{sp},W\otimes F'} |\Phi^{-1}|_{\mathrm{sp},W\otimes F'} \leq s s^{-1} = 1,$$

which forces $|\Phi|_{\mathrm{sp},W} = |\Phi|_{\mathrm{sp},W\otimes F'} = s$. □

Corollary 9. *Let V_1, V_2 be finite difference modules over F which are pure of respective norms s_1, s_2. Then $V_1 \otimes_F V_2$ is pure of norm $s_1 s_2$.*

Proof. If $s_1 s_2 = 0$, then it is easy to check that $V_1 \otimes V_2$ is pure of norm 0. Otherwise, one direction of Proposition 8 yields

$$|\Phi|_{\mathrm{sp},V_1\otimes V_2\otimes F'} = s_1 s_2, \qquad |\Phi^{-1}|_{\mathrm{sp},V_1\otimes V_2\otimes F'} = s_1^{-1} s_2^{-1},$$

so the other direction of Proposition 8 implies that $V_1 \otimes V_2$ is pure of norm $s_1 s_2$. □

Corollary 10. *Let V be a finite difference module over F. Then for any positive integer d, V is pure of norm s if and only if V becomes pure of norm s^d when viewed as a difference module over (F, ϕ^d).*

Proposition 11. *Suppose that either:*

(a) $|\Phi|_{\mathrm{sp},V} < 1$, or

(b) F is inversive and $|\Phi^{-1}|_{\mathrm{sp},V} < 1$.

Then $H^1(V) = 0$.

Proof. In case (a), given $v \in V$, the series

$$w = \sum_{i=0}^{\infty} \Phi^i(v)$$

converges to a solution of $w - \Phi(w) = v$. In case (b), the series

$$w = -\sum_{i=0}^{\infty} \Phi^{-i-1}(v)$$

does likewise. □

Corollary 12. *If V_1, V_2 are finite differential modules pure of norms s_1, s_2, and either:*

(a) $s_1 < s_2$; or

(b) F is inversive and $s_1 > s_2$;

then any exact sequence $0 \to V_1 \to V \to V_2 \to 0$ splits.

Proof. If $s_2 > 0$, then by Corollary 9, $V_2^\vee \otimes V_1$ is pure of norm s_1/s_2, so Proposition 11 gives the desired splitting. Otherwise, we must be in case (b), so we can pass to the opposite ring to make the same conclusion. $\square$

If F is inversive, we again get a decomposition theorem.

Theorem 13. *Suppose that F is inversive, and let V be a finite difference module over F. Then there exists a unique direct sum decomposition*

$$V = \bigoplus_{s \geq 0} V_s$$

of difference modules, in which each V_s is pure of norm s. (Note that V is dualizable if and only if $V_0 = 0$.)

Proof. This follows at once from Corollary 12. $\square$

Note that in case ϕ is trivial, this simply reproduces the decomposition of V in which the generalized eigenspaces for all eigenvalues of a given modulus are grouped together.

If F is not inversive, we only get a filtration.

Theorem 14. *Let V be a finite difference module over F. Then there exists a unique filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V$$

of difference modules, such that each successive quotient V_i/V_{i-1} is pure of some norm s_i, and $s_1 > \cdots > s_l$. (Note that V is dualizable if and only if $V = 0$ or $s_l > 0$.)

Proof. Start with any filtration of V with irreducible successive quotients, and let s_1 be the largest norm which appears. By Corollary 12, we can change the filtration to move the first appearance of s_1 one step earlier; consequently, we can put all appearances of s_1 before all other slopes. Group these together to form V_1, then repeat to construct the desired factorization. Uniqueness follows by tensoring with F' and invoking the uniqueness in Theorem 13. $\square$

The following alternate characterization of pureness may be useful in some situations.

Proposition 15. *Let V be a finite difference module over F, and choose $\lambda \in F^\times$. Then V is pure of norm $|\lambda|$ if and only if there exists a basis of V on which Φ acts via λ times an element of $\mathrm{GL}_n(\mathfrak{o}_F)$.*

Note that whenever V is pure of positive norm, we can apply this result after replacing Φ by some power of it, thanks to Corollary 7.

Proof. If such a basis exists, then Proposition 8 implies that V is pure of norm $|\lambda|$. Conversely, if V is irreducible of spectral norm $|\lambda|$, then Lemma 6 provides a basis of the desired form. Otherwise, we proceed by induction on $\dim_F V$. Suppose we are given a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ in which V_1, V_2 admit bases of the desired form. Let $e_1, \dots, e_m \in V$ form such a basis for V_1, and let $e_{m+1}, \dots, e_n \in V$ lift such a basis for V_2. Then for $\mu \in F$ of sufficiently small norm,

$$e_1, \dots, e_m, \mu e_{m+1}, \dots, \mu e_n$$

will form a basis of V of the desired form. □

5 Hodge and Newton polygons

Let V be a finite difference module over F equipped with a norm defined as the supremum norm for some basis $e_1, \dots, e_n$. Let A be the basis via which Φ acts on this basis; define the *Hodge polygon* of V as the Hodge polygon of the matrix A. Given the choice of the norm on V, this definition is independent of the choice of the basis: we can only change basis by a matrix $U \in \mathrm{GL}_n(\mathfrak{o}_F)$, which replaces A by $U^{-1}A\phi(U)$, and ϕ being an isometry ensures that $\phi(U) \in \mathrm{GL}_n(\mathfrak{o}_F)$ also. As in the linear case, we list the Hodge slopes $s_{H,i}, \dots, s_{H,n}$ in increasing order.

Define the *Newton polygon* of V to have slopes $s_{N,1}, \dots, s_{N,n}$ such that r appears with multiplicity equal to the dimension of the quotient in Theorem 14 of norm e^{-r}.

Lemma 16. *Let V be a finite difference module over F. We have*

$$\begin{aligned} s_{H,1} + \cdots + s_{H,i} &= -\log |\Phi|_{\wedge^i V} \qquad (i = 1, \dots, n) \\ s_{N,1} + \cdots + s_{N,i} &= -\log |\Phi|_{\mathrm{sp}, \wedge^i V} \qquad (i = 1, \dots, n). \end{aligned}$$

Proof. The first assertion follows from the corresponding fact in the linear case. The second assertion reduces to the fact that if V is irreducible of dimension n and spectral norm s, then $\wedge^i V$ has spectral norm s^i for $i = 1, \dots, n$; this may be read off from the basis used in the proof of Lemma 6. □

Corollary 17 (Newton above Hodge)**.** *We have*

$$s_{N,1} + \cdots + s_{N,i} \geq s_{H,1} + \cdots + s_{H,i} \qquad (i = 1, \dots, n)$$

with equality for $i = n$.

Beware that the Newton polygon, unlike the Hodge polygon, cannot be directly read off from the matrix via which Φ acts on some basis; see exercises for a counterexample. On the other hand, this works if the matrix of Φ is a *companion matrix*, i.e., a matrix of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}$$

with 1s on the subdiagonal, arbitrary entries in the last column, and zeroes elsewhere; this is a restatement of the following fact.

Proposition 18. *If $V \cong F\{T\}/F\{T\}P$, then the Newton polygon of V coincides with that of P.*

Proof. This reduces to Lemma 6. □

6 The Dieudonné-Manin classification theorem

For $\lambda \in F$ and d a positive integer, let $V_{\lambda,d}$ be the difference module over F with basis $e_1, \dots, e_d$ such that

$$\Phi(e_1) = e_2, \quad \dots, \quad \Phi(e_{d-1}) = e_d, \quad \Phi(e_d) = \lambda e_1.$$

Lemma 19. *Suppose $\lambda \in F^\times$ and the positive integer d are such that there is no $i \in \{1, \dots, d-1\}$ such that $|\lambda|^{i/d} \in |F^\times|$. Then $V_{\lambda,d}$ is irreducible.*

Proof. Note that

$$\Phi^d e_i = \phi^{i-1}(\lambda) e_i \qquad (i = 1, \dots, n).$$

Hence by Proposition 15, $V_{\lambda,d}$ is pure of norm $\lambda^{1/d}$, as then is any submodule. But if the submodule were proper and nonzero, we would have a violation of Corollary 7. □

Theorem 20. *Let F be a complete discretely valued field equipped with an isometric endomorphism ϕ, such that κ_F is strongly difference-closed. Then every dualizable finite difference module over F can be split (non-uniquely) as a direct sum of submodules, each of the form $V_{\lambda,d}$ for some λ, d. Moreover, for π any fixed uniformizer of F, we can force each λ to be a power of π.*

Proof. We first check that if V is pure of norm 1, then V is trivial. We must show that for any $A \in \mathrm{GL}_n(\mathfrak{o}_F)$, there exists a convergent sequence $U_1, U_2, \dots \in \mathrm{GL}_n(\mathfrak{o}_F)$ such that

$$U_m^{-1} A \phi(U_m) \equiv I_n \pmod{\pi^m}.$$

Specifically, we will insist that $U_{m+1} \equiv U_m \pmod{\pi^m}$. Finding U_1 amounts to trivializing a dualizable difference module of dimension m over κ_F. For $m > 1$, given U_m, we must have $U_{m+1} = U_m(I_n + \pi^m X_m)$ for some m, and

$$(I_n + \pi^m X_m)^{-1}(U_m^{-1} A\phi(U_m))(I_n + \pi^m X_m) \equiv I_n \pmod{\pi^{m+1}}.$$

Since already $U_m^{-1} A\phi(U_m) \equiv I_n \pmod{\pi^m}$, this amounts to solving

$$-X_m + \pi^{-m}(U_m^{-1} A\phi(U_m) - I_n) + \phi(X_m) \equiv 0 \pmod{\pi},$$

which we solve by applying criterion (c) from Lemma 3.

By similar (but easier) arguments, we also show that:

- ϕ is surjective on $\mathfrak{o}_F$, so F is inversive;
- if V is trivial, then $H^1(V) = 0$.

In particular, we may apply Theorem 13 to reduce the desired result to the case where V is pure of norm $s > 0$.

Let d be the smallest positive integer such that $s^d = |\pi^m|$ for some integer m. Then the first paragraph implies that $\pi^{-m}\Phi^d$ fixes some nonzero element of V; this gives us a nonzero map from $V_{\pi^m,d}$ to V. By Lemma 19, this map must be injective. Repeating this argument, we write V as a successive extension of copies of $V_{\pi^m,d}$. However, $V_{\pi^m,d}^\vee \otimes V_{\pi^m,d}$ is pure of norm 1, so has trivial H^1 as above. Thus V splits as a direct sum of copies of $V_{\pi^m,d}$, as desired. □

By Proposition 4, Theorem 20 has the following immediate corollary.

Corollary 21. *Let F be a complete discretely valued field, normalized so that the additive value group is $\mathbb{Z}$, such that κ_F is algebraically closed of characteristic $p > 0$. Let $\phi : F \to F$ be an isometric automorphism lifting a power of the absolute Frobenius on κ_F. Then every dualizable finite difference module over F can be split (non-uniquely) as a direct sum of difference submodules, each of the form $V_{\lambda,d}$ for some $\lambda \in F^\times$ and some positive integer d coprime to the valuation of λ. Moreover, for π any fixed uniformizer of F, we can force each λ to be a power of π.*

The case in which k is an algebraically closed field of characteristic p, $W(k)$ is the ring of p-typical Witt vectors (i.e., the unique complete discrete valuation ring with residue field k and maximal ideal (p)), $F = \mathrm{Frac}(W(k))$, and ϕ is the Witt vector Frobenius is the *Dieudonné-Manin theorem*, i.e., the classification theorem of rational Dieudonné modules over an algebraically closed field.

7 Notes

The parallels between difference and differential algebra are quite close, enough so that a survey of references for difference algebra strongly resembles its differential counterpart. An older, rather dry reference is [Coh65]; a somewhat more lively modern reference, which develops difference Galois theory under somewhat restrictive conditions, is [SvdP97]. We again mention [And01] as a unifying framework for difference and differential algebra.

In the special case of the difference field $\mathrm{Frac}(W(k))$, with k perfect of characteristic $p > 0$, most of the results of this section appear in [Kat79] in some form, but it is awkward to give direct references since we have organized our presentation rather differently.

Proposition 4 can be found in SGA7 [DK73, Exposé XXII, Corollaire 1.1.10], wherein Katz attributes it to Lang. Indeed, it is a special case of the nonabelian Artin-Schreier theory associated to an algebraic group over a field of positive characteristic (in our case GL_n), via the *Lang torsor*; see [Lan56].

For the original classification of rational Dieudonné modules over an algebraically closed field, see Manin's original paper [Man63] or the book of Demazure [Dem72].

8 Exercises

1. Let F be a difference field of characteristic zero containing an element x such that $\phi(x) = \lambda x$ for some λ fixed by ϕ. Prove that every finite difference module for M admits a cyclic vector. (Hint: under these hypotheses, one can readily imitate the proof of the cyclic vector theorem for differential modules.)

2. Let F be the completion of $\mathbb{Q}_p(t)$ for the 1-Gauss norm, viewed as a difference field for ϕ equal to the substitution $t \mapsto t^p$. Let V be the difference module corresponding to the matrix
$$A = \begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}.$$
Prove that there is a nonsplit short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with V_1, V_2 pure of slopes s_1, s_2 with $s_1 < s_2$.

3. Here is a beautiful example from [Kat79, §1.3] (attributed to B. Gross). Let p be a prime congruent to 3 modulo p, put $F = \mathbb{Q}_p(i)$ with $i^2 = -1$, and let ϕ be the automorphism $i \mapsto -i$ of F over $\mathbb{Q}_p$. Define a difference module M of rank 2 over F using the matrix
$$A = \begin{pmatrix} 1-p & (p+1)i \\ (p+1)i & p-1 \end{pmatrix}.$$
Compute the Newton polygons of A and M and verify that they do not coincide. (Hint: find another basis of M on which Φ acts diagonally.)

4. Prove that every difference field can be embedded into a difference-closed field. (This requires your favorite equivalent of the axiom of choice, e.g., Zorn's lemma.)