p-adic differential equations
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Discs and annuli

In this unit, we introduce p-adic closed discs and annuli, but in a purely ring-theoretic fashion. This avoids having to introduce any p-adic analytic geometry.

Throughout the unit (and in all later units, unless explicitly contravened), let K be a field complete for a nontrivial nonarchimedean valuation $|\cdot|$. Assume that K has characteristic 0, but the residue field κ_K has characteristic $p > 0$. Also assume that things are normalized so that $|p| = p^{-1}$.

1 Power series on closed discs and annuli

We start by introducing some rings that should be thought of as the analytic functions on a closed disc $|t| \leq \beta$, or a closed annulus $\alpha \leq |t| \leq \beta$. As noted in the introduction, this is more properly done in a framework of p-adic analytic geometry, but we will avoid this framework.

For $\alpha, \beta > 0$, put

$$K\langle \alpha/t, t/\beta \rangle = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i \in K[\![t, t^{-1}]\!] : \lim_{i \to \pm\infty} |c_i| \rho^i = 0 \quad (\rho \in [\alpha, \beta]). \right\}.$$

That is, consider formal bidirectional power series which converge whenever you plug in a value for t with $|t| \in [\alpha, \beta]$, or in other words, when $\alpha/|t|$ and $|t|/\beta$ are both at most 1; it suffices to check for $\rho = \alpha$ and $\rho = \beta$. Although formal bidirectional power series do not form a ring, the subset $K\langle \alpha/t, t/\beta \rangle$ does form a ring under the expected operations.

If $\alpha = 0$, the only reasonable interpretation of the previous definition is to require $c_i = 0$ for $i < 0$. When there are no negative powers of t, it is redundant to require the convergence for $\rho < \beta$. In other words,

$$K\langle 0/t, t/\beta \rangle = K\langle t/\beta \rangle = \left\{ \sum_{i=0}^{\infty} c_i t^i \in K[\![t]\!] : \lim_{i \to \infty} |c_i| \beta^i = 0 \right\}.$$

One could also allow $\beta = \infty$ for a similar effect in the other direction. More succinctly put, we identify $K\langle \alpha/t, t/\beta \rangle$ with $K\langle \beta^{-1}/t^{-1}, t^{-1}/\alpha^{-1} \rangle$.

2 Gauss norms and Newton polygons

The rings $K\langle \alpha/t, t/\beta \rangle$ quite a lot like polynomial rings (or Laurent polynomial rings, in case $\alpha \neq 0$) in one variable. The next few statements are all instances of this analogy.

From the definition of $K\langle \alpha/t, t/\beta \rangle$, we see that it carries a well-defined ρ-Gauss norm

$$\left| \sum_i c_i t^i \right|_\rho = \max_i \{ |c_i| \rho^i \}$$

for any $\rho \in [\alpha, \beta]$. For $\rho = \alpha = 0$, this reduces to simply $|c_0|$.

The additive version is this is to take $r \in [-\log \beta, -\log \alpha]$ and put

$$v_r \left(\sum c_i t^i \right) = \min_i \{ v(c_i) + ri \},$$

where $v(c) = -\log |c|$. This is the same formula as we had for the sloped valuation function on a polynomial ring, so we may repeat the proof to obtain the following.

Lemma 1. *For $r \in [-\log \beta, -\log \alpha]$, the function v_r on $K\langle \alpha/t, t/\beta \rangle$ is a valuation; in particular, $v_r(xy) = v_r(x) + v_r(y)$. Equivalently, for $\rho \in [\alpha, \beta]$, the ρ-Gauss norm on $K\langle \alpha/t, t/\beta \rangle$ is really a norm; that is, it indeed satisfies $|fg|_\rho = |f|_\rho |g|_\rho$.*

One may define the *Newton polygon* for an element $x = \sum x_i t^i \in K\langle \alpha/t, t/\beta \rangle$ as the lower convex hull of the set

$$\{(-i, v(x_i)) : i \in \mathbb{Z}, x_i \neq 0\},$$

except that we only keep the slopes in $[-\log \beta, -\log \alpha]$.

Proposition 2. *Let $x = \sum_i x_i t^i \in K\langle \alpha/t, t/\beta \rangle$ be nonzero.*

(a) The Newton polygon of x has finite width.

(b) The function $r \mapsto v_r(x)$ on $[-\log \beta, -\log \alpha]$ is continuous, piecewise affine, and convex.

(c) The function $\rho \mapsto |x|_\rho$ on $[\alpha, \beta]$ is continuous and log-concave. The log-concavity means that $\rho, \sigma \in [\alpha, \beta]$ and $c \in [0, 1]$, put $\tau = \rho^c \sigma^{1-c}$; then

$$|x|_\tau \leq |x|_\rho^c |x|_\sigma^{1-c}.$$

(d) If $\alpha = 0$, then v_r is decreasing on $[-\log \beta, \infty)$; in other words, for all $\rho \in [0, \beta]$, $|x|_\rho \leq |x|_\beta$.

Part (c) should be thought of as a nonarchimedean analogue of the Hadamard three circle theorem.

Proof. We have (a) because there is a least i for which $|c_i|\alpha^i$ is maximized, and there is a greatest j for which $|c_j|\beta^j$ is maximized. This implies (b) because as in the polynomial case, we may interpret $v_r(x)$ as the y-intercept of the supporting line of the Newton polygon of slope r. This in turn implies (c), and (d) is a remark made earlier. □

When dealing with the ring $K\langle \alpha/t, t/\beta\rangle$, the following completeness property will be extremely useful.

Proposition 3. *The ring $K\langle \alpha/t, t/\beta\rangle$ is Fréchet complete for the norms $|\cdot|_\rho$ for all $\rho \in I$. That is, if $\{x_n\}_{n=0}^\infty$ is a sequence which is simultaneously Cauchy under $|\cdot|_\rho$ for all $\rho \in I$, then it is convergent. (By Proposition 2, it suffices to check the Cauchy property at each nonzero endpoint of I.)*

Proof. Exercise. □

For instance, the completeness property is used in the construction of multiplicative inverses.

Lemma 4. *If $\alpha = 0$ (resp. $\alpha > 0$), a nonzero element $f \in K\langle \alpha/t, t/\beta\rangle$ is a unit if and only if v_r is constant (resp. affine) on $[-\log(\beta), -\log(\alpha)]$.*

Proof. We will just consider the case $\alpha > 0$; the other case is similar (and easier). Put $f = \sum_{i\in\mathbb{Z}} f_i t^i$. Note that the following are equivalent:

(a) there is a single i for which $|f|_\rho = |f_i|\rho^i$ for all $\rho \in [\alpha, \beta]$;

(b) the function $r \mapsto v_r(f)$ on $[-\log(\beta), -\log(\alpha)]$ is affine;

(c) the Newton polygon of f has no slopes in $[-\log(\beta), -\log(\alpha)]$.

By (c), these conditions all hold if f is a unit. Conversely, if these conditions hold, then the series

$$(f_i t_i)^{-1}(1 - (f_i t^i - f)/(f_i t^i))^{-1} = \sum_{j=0}^{\infty} (f_i t^i - f)^j (f_i t_i)^{-j-1}$$

converges by Proposition 3, and its limit is an inverse of f. □

3 Factorization results

Proposition 5 (Weierstrass preparation)**.** *Suppose that $f = \sum_{i\in\mathbb{Z}} f_i t^i \in K\langle \alpha/t, t/\beta\rangle$, and that $\rho \in [\alpha, \beta]$ is such that there is a unique $m \in \mathbb{Z}$ maximizing $|f_m|\rho^m$. Then there is a unique factorization $f = f_m t^m gh$ with*

$$g \in K\langle \alpha/t, t/\beta\rangle \cap K[\![t]\!] = K\langle t/\beta\rangle,$$
$$h \in K\langle \alpha/t, t/\beta\rangle \cap K[\![t^{-1}]\!] = K\langle \alpha/t\rangle,$$

$|g|_\rho = |g_0| = 1$, *and* $|h - 1|_\rho < 1$.

Proof. The master slope factorization applies thanks to Property 3. □

In light of the finite width property of the Newton polygon, the following should not be a surprise.

Proposition 6 (More Weierstrass preparation)**.** *For $f \in K\langle \alpha/t, t/\beta \rangle$, there exists a polynomial $P \in K[t]$ and a unit $g \in K\langle \alpha/t, t/\beta \rangle^\times$ such that $f = Pg$. In particular, $K\langle \alpha/t, t/\beta \rangle$ is a principal ideal domain.*

Proof. Using Proposition 5, we may reduce to two instances of the case $\alpha = 0$, so we restrict to that case hereafter. Put $f = \sum_i f_i t^i$, and choose m maximizing $|f_m|\beta^m$. Let R be the ring of formal sums $\sum_i c_i t^i$ of series with $|c_i|\beta^i$ bounded as $i \to -\infty$ and tending to 0 as $i \to +\infty$. Let e be the inverse of $\sum_{i=0}^m f_i t^i$ in R, and apply master slope factorization to factor $ef = gh$ in R, in which g is a unit in $K\langle t/\beta \rangle$ by Lemma 4. Now $h \sum_{i=0}^m f_i t^i = fg^{-1}$ belongs to

$$K[\![t]\!] \cap t^m K[\![t^{-1}]\!].$$

It is thus a polynomial of degree m, proving the claim. □

We will make frequent and often implicit use of the following patching lemma.

Lemma 7 (Patching lemma)**.** *Suppose $\alpha \leq \gamma \leq \beta \leq \delta$. Let M_1 be a finite free module over $K\langle \alpha/t, t/\beta \rangle$, let M_2 be a finite free module over $K\langle \gamma/t, t/\delta \rangle$, and suppose we are given an isomorphism*

$$\psi : M_1 \otimes K\langle \gamma/t, t/\beta \rangle \cong M_2 \otimes K\langle \gamma/t, t/\beta \rangle.$$

Then we can find a finite free module M over $K\langle \alpha/t, t/\delta \rangle$ and isomorphisms $M_1 \cong M \otimes K\langle \alpha/t, t/\beta \rangle$, $M_2 \cong M \otimes K\langle \gamma/t, t/\delta \rangle$ inducing ψ. Moreover, M is determined by this requirement up to unique isomorphism.

Proof. We will only explain the case $\alpha > 0$; the case $\alpha = 0$ is similar.

Choose bases of M_1 and M_2 and let A be the $n \times n$ matrix defining ψ; then A must be invertible over $K\langle \gamma/t, t/\beta \rangle$. Choose $\rho \in [\gamma, \beta]$; since $\det(A)$ is a unit in $K\langle \gamma/t, t/\beta \rangle$, we can find an invertible $n \times n$ matrix W over $K\langle \gamma/t, t/\beta \rangle$ such that $\det(WA) = 1$. (For instance, take $W = \mathrm{Diag}(\det(A)^{-1}, 1, \ldots, 1)$.)

It is then possible (see exercises) to find invertible matrices U, V over $K[t, t^{-1}]$ such that $|UWAV - I_n|_\rho < 1$. By changing the initial choices of bases, we can force ourselves into the case $|A - I_n|_\rho < 1$.

By using the master slope factorization in the matrix ring over $K\langle \gamma/t, t/\beta \rangle$, we can split A as a product of an invertible matrix over $K\langle t/\beta \rangle$ and an invertible matrix over $K\langle \gamma/t \rangle$. Using these to change basis in M_1 and M_2, respectively, we can put ourselves in the situation where $A = I_n$, in which case we may identify the bases of M_1 and M_2. Take M to be the free module over $K\langle \alpha/t, t/\delta \rangle$ with the same basis. □

4 Notes

The Hadamard three circles theorem (Proposition 2(c)) is a special case of the fact that the *Shilov boundary* of the annulus $\alpha \leq |t| \leq \beta$ consists of the two circles $|t| = \alpha$ and $|t| = \beta$. For much amplification of this remark, including a full-blown theory of harmonic functions on Berkovich analytic curves, see [Thu05]. For an alternate presentation, restricted to the Berkovich projective line but otherwise more detailed, see [BR07].

The patching lemma (Lemma 7) is a special case of the glueing property of coherent sheaves on affinoid rigid analytic spaces, i.e., the theorems of Kiehl and Tate [BGR84, Theorems 8.2.1/1 and 9.4.2/3]. The factorization argument in the proof, however, is older still; it is the nonarchimedean version of what is called a *Birkhoff factorization* over an archimedean field.

5 Exercises

1. Prove Proposition 3. (Hint: it may be easiest to first construct the limit using a single $\rho \in [\alpha, \beta]$, then show that it must also work for the other ρ.)

2. Let R be the ring of formal power series over K which converge for $|t| < 1$. Prove that R is not noetherian; this is why I avoided introducing it.

3. Suppose K is complete for a discrete valuation. Prove that any element of $\mathfrak{o}_K[\![t]\!] \otimes_{\mathfrak{o}_K} K$ (that is, a power series with bounded coefficients) is equal to a polynomial in t times a unit. Then prove that this fails if K is complete for a nondiscrete valuation.

4. Let A be an $n \times n$ matrix over $K\langle \rho/t, t/\rho \rangle$ such that $|\det(A) - 1|_\rho < 1$. Prove that there exist invertible matrices U, V over $K[t, t^{-1}]$ such that $|U^{-1}AV - I_n|_\rho < 1$. (Hint: perform approximate Gaussian elimination. An analogous argument, but in more complicated notation, is [Ked04, Lemma 6.2]. We will see a similar result in the unit on numerical analysis.)