CHAPTER 16

# Effective convergence bounds

In this chapter, we discuss some effective bounds on the solutions of $p$-adic differential equations with nilpotent singularities; we put this chapter here partly to illustrate the improvement one gets in the bounds by accounting for a Frobenius structure. Just like their archimedean counterparts, these are important for carrying out rigorous numerical calculations.

## 1. Nilpotent singularities in the $p$-adic setting

For applications in geometry, it is important to have effective bounds not just for nonsingular differential equations, but also for some regular singular differential equations. However, in the $p$-adic case, the $p$-adic behavior of the exponents creates many headaches. The case where the exponents are all zero is an important middle ground.

PROPOSITION 16.1.1. *Let $N = \sum_{i=0}^{\infty} N_i t^i$ be an $n \times n$ matrix over $K\langle t/\beta \rangle$ corresponding to the differential system $D(v) = Nv + d(v)$, where $d = t\frac{d}{dt}$. Assume that $N_0$ is nilpotent with nilpotency index $m$; that is, $N_0^m = 0$ but $N_0^{m-1} \neq 0$. Assume also that $|N_0| \leq 1$. Then the fundamental solution matrix $U = \sum_{i=0}^{\infty} U_i t^i$ over $K[\![t]\!]$ (as in Proposition 6.3.4) satisfies*

$$|U_i| \leq |i!|^{-2m+1} \max\{|N_j| : 0 \leq j \leq i\} \qquad (i = 1, 2, \dots). \tag{16.1.1.1}$$

*Consequently, $U$ has entries in $K\langle t/(p^{-(2m-1)/(p-1)}\beta)\rangle$ (as does its inverse).*

Note that this reproves the $p$-adic Cauchy theorem (Proposition 8.2.3).

PROOF. Recall (6.3.4.1):

$$N_0 U_i - U_i N_0 + iU_i = -\sum_{j=0}^{i-1} N_{i-j} U_j.$$

The map $f(X) = NX - XN$ on $n \times n$ matrices is nilpotent with nilpotency index $2m - 1$. Hence the inverse of the map $X \mapsto iX + f(X)$ has inverse

$$X \mapsto \sum_{j=0}^{2m-2} (-1)^j i^{-j-1} f^j(X).$$

This gives the claim by induction on $i$. ∎

## 2. Effective bounds for solvable modules

We now give an improved version of Proposition 16.1.1 under the hypothesis that $U$ has entries in $K\langle t/\beta \rangle$. The hypothesis is only qualitative, in that it implies that $|U_i|\beta^i \to 0$ as $i \to \infty$ but does not give a specific bound on $|U_i|$ for any particular $i$. Somewhat surprisingly, this hypothesis plus any explicit bound on $N$ together imply a rather strong explicit bound on $|U_i|$. We first suppose the bound on $N$ is of a specific form.

THEOREM 16.2.1. *Let $N = \sum_{i=0}^{\infty} N_i t^i$, $U = \sum_{i=0}^{\infty} U_i t^i$ be $n \times n$ matrices over $K[\![t]\!]$ such that*

(a) *$N$ has entries in $K\langle t/\beta \rangle$;*
(b) *$U_0 = I_n$;*
(c) *$U^{-1}NU + U^{-1}t\frac{d}{dt}(U) = N_0$;*
(d) *$N_0$ is nilpotent;*
(e) *$U$ and $U^{-1}$ have entries in $K\langle t/\beta \rangle$.*

*Then for every nonnegative integer $i$,*

$$|U_i|\beta^i \leq p^{(n-1)\lfloor \log_p i \rfloor} \max\{1, |N|_\beta^{n-1}\}.$$

The first step in the proof of Theorem 16.2.1 is to change basis to reduce $|N|_\beta$; however, we pay the price of decreasing $\beta$ slightly.

LEMMA 16.2.2. *With notation as in Theorem 16.2.1, for any $\lambda < 1$, $\mu > 1$, there exists an invertible $n \times n$ matrix $X = \sum_{i=0}^{\infty} U_i t^i$ over $K\langle t/(\lambda\beta) \rangle$ such that*

$$\begin{aligned} |X^{-1}NX + U^{-1}t\frac{d}{dt}(X)|_{\lambda\beta} &\leq 1 \\ |X^{-1}|_{\lambda\beta} &\leq \mu \\ |X|_{\lambda\beta} &\leq |N|_\beta^{n-1}\mu. \end{aligned}$$

PROOF. Let $M$ be the differential module over $K\langle t/\beta \rangle$ for the operator $t\frac{d}{dt}$, with a basis on which $D$ acts via $N$, and let $|\cdot|$ be the supremum norm defined by this basis. Since the fundamental solution matrix for $M$ converges in the closed disc of radius $\beta$, $M$ has generic radius of convergence $\beta$. In particular,

$$|D|_{\mathrm{sp},M} \leq \left| t\frac{d}{dt} \right|_{F_\beta} = 1.$$

By Proposition 5.2.11 plus the lattice lemma (Lemma 10.5.1), for any desired $\epsilon > 0$, we may find $V \in \mathrm{GL}_n(K\langle t/(\lambda^{1/2}\beta) \rangle)$ such that for $N' = V^{-1}NV + V^{-1}t\frac{d}{dt}(V)$,

$$\begin{aligned} |N'|_\beta &\leq 1 + \epsilon \\ |V^{-1}|_\beta &\leq 1 + \epsilon \\ |V|_\beta &\leq |N|_\beta^{n-1}(1 + \epsilon). \end{aligned}$$

Since the constant coefficient $N'_0$ of $N'$ is nilpotent, it has spectral norm 0. By Proposition 3.4.6, there exists $W \in \mathrm{GL}_n(K)$ with

$$|W^{-1}| \leq 1, \qquad |W| \leq (1+\epsilon)^{n-1}, \qquad |W^{-1}N'_0 W| \leq 1.$$

We now take $X = VW$, so that $X^{-1}NX + X^{-1}t\frac{d}{dt}(X) = W^{-1}N'W$. We then have

$$
\begin{aligned}
|(W^{-1}N'W)_0| &\leq 1 \\
|W^{-1}N'W|_\beta &\leq (1+\epsilon)^n \\
|X^{-1}|_\beta &\leq 1+\epsilon \\
|X|_\beta &\leq |N|_\beta^{n-1}(1+\epsilon)^n.
\end{aligned}
$$

For $\epsilon$ such that $(1+\epsilon)^n \leq \max\{\lambda^{-1}, \mu\}$, we have the desired inequalities. $\square$

Using Lemma 16.2.2, we prove Theorem 16.2.1 by using Frobenius antecedents to reduce the index from $i$ to $\lfloor i/p \rfloor$. One can improve upon this argument if one has a Frobenius structure on the differential module; see Lemma 16.3.2.

Lemma 16.2.3. *With notation as in Theorem 16.2.1, suppose that $|N|_\beta \leq 1$. Then for any $\lambda < 1$, $\mu > 1$, there exist $n \times n$ matrices $N', U'$ over $K\langle t/(\lambda^p\beta^p)\rangle$ satisfying the hypotheses of Theorem 16.2.1, such that*

$$
\begin{gathered}
|N'|_{\lambda\beta} \leq p \\
\max\{|U_j|(\lambda\beta)^j : 0 \leq j \leq i\} \leq \max\{|U'_j|(\lambda\beta)^{pj} : 0 \leq j \leq i/p\}.
\end{gathered}
$$

Proof. Define the invertible $n \times n$ matrix $V = \sum_{i=0}^\infty V_i t^i$ over $K[\![t]\!]$ as follows. Start with $V_0 = I_n$. Given $V_0, \dots, V_{i-1}$, if $i \equiv 0 \pmod p$, put $V_i = 0$. Otherwise, put $W = \sum_{j=0}^{i-1} V_j t^j$ and $N_W = W^{-1}NW + W^{-1}t\frac{d}{dt}(W)$, and let $V_i$ be the unique solution of the matrix equation

$$N_0 V_i - V_i N_0 + iV_i = -(N_W)_i.$$

By induction on $i$, $|V_i|\beta^i \leq 1$ for all $i$, so $V$ is invertible over $K\langle t/(\lambda^{1/2}\beta)\rangle$.

Let $\phi : K[\![t]\!] \to K[\![t]\!]$ denote the substitution $t \mapsto t^p$. Put $N'' = V^{-1}NV + V^{-1}t\frac{d}{dt}(V)$; then $N''$ has entries in $K[\![t^p]\!]$, and $|\phi^{-1}(N'')|_{\lambda^{p/2}\beta^p} \leq 1$. Put $U'' = V^{-1}U$, so that $|U''|_{\lambda^{1/2}\beta} = 1$; then

$$(U'')^{-1}N''U'' + (U'')^{-1}t\frac{d}{dt}(U'') = N''_0 = N_0,$$

which forces $U''$ also to have entries in $K[\![t^p]\!]$. We may then take $N' = p^{-1}\phi^{-1}(N'')$ and $U' = \phi^{-1}(U'')$. $\square$

We now put everything together.

Proof of Theorem 16.2.1. We prove the claim by induction on $i$, in three stages. First, if $i < p$ and $|N|_\beta \leq 1$, then the desired estimate follows from Proposition 16.1.1. Second, for any given $i$, the desired estimate for general $N$ follows from the estimate for the same $i$ in the case $|N|_\beta \leq 1$, by Lemma 16.2.2. (More precisely, for any $\lambda < 1, \mu > 1$, replace the pair $N, U$ by $X^{-1}NX + X^{-1}t\frac{d}{dt}(X), X^{-1}UX_0$; then take the limit as $\lambda, \mu \to 1$.) Third, if $|N|_\beta \leq 1$, then the desired estimate for any given $i$ follows from the corresponding estimate for general $N$ with $i$ replaced by $\lfloor i/p \rfloor$, by Lemma 16.2.3 (again applying the argument for any $\lambda < 1, \mu > 1$, then taking the limit as $\lambda, \mu \to 1$). $\square$

We will often apply Theorem 16.2.1 through the following corollary (deduced by taking $\beta$ to be an arbitrary value less than 1).

THEOREM 16.2.4. *Let* $N = \sum_{i=0}^\infty N_i t^i$, $U = \sum_{i=0}^\infty U_i t^i$ *be* $n \times n$ *matrices over* $K[\![t]\!]$ *such that:*

(a) $|N|_1 < \infty$ *(i.e.,* $|N_i|$ *is bounded over all* $i$*);*
(b) $U_0 = I_n$*;*
(c) $U^{-1}NU + U^{-1}t\frac{d}{dt}(U) = N_0$*;*
(d) $N_0$ *is nilpotent;*
(e) *for all* $\beta < 1$, $U$ *and* $U^{-1}$ *have entries in* $K\langle t/\beta \rangle$*.*

*Then for every nonnegative integer* $i$*,*

$$|U_i| \leq p^{(n-1)\lfloor \log_p i \rfloor} |N|_1^{n-1}.$$

EXAMPLE 16.2.5. It is easy to make an example that shows that one cannot significantly improve the bound of Theorem 16.2.1 without extra hypotheses. (There is a tiny improvement possible; see notes.) For instance, one can use the functions

$$f_i = \frac{1}{i!}(\log(1+t))^i \qquad (i = 0, \dots, n-1)$$

which satisfy the differential system

$$\frac{d}{dt} f_0 = 0, \quad \frac{d}{dt} f_i = \frac{1}{1+t} f_{i-1} \qquad (i = 1, \dots, n-1),$$

in which the coefficients have 1-Gauss norm at most 1.

## 3. Frobenius structures

Although Theorem 16.2.4 is close to optimal under its hypotheses, it can be improved in case the differential module in question admits a Frobenius structure.

HYPOTHESIS 16.3.1. In this section, fix a power $q$ of $p$, and let $\phi$ be a scalar-centered $q$-power Frobenius lift on $K[\![t]\!]_0$.

The key here is to imitate the proof of Theorem 16.2.1 with the differential equation replaced by a certain Frobenius equation.

LEMMA 16.3.2. *Let* $U = \sum_{i=0}^\infty U_i t^i$, $A = \sum_{i=0}^\infty A_i t^i$ *be* $n \times n$ *matrices over* $K[\![t]\!]$ *such that:*

(a) $|A|_1 < \infty$*;*
(b) $U_0 = I_n$ *and* $A_0$ *is invertible;*
(c) $U^{-1}A\phi(U) = A_0$*.*

*Then*

$$\max\{|U_j| : 0 \leq j \leq i\} \leq |A|_1 |A_0^{-1}| \max\{|U_j| : 0 \leq j \leq i/q\}.$$

*Consequently, for every nonnegative integer* $i$*,*

$$|U_i| \leq (|A|_1 |A_0^{-1}|)^{\lceil \log_q i \rceil}.$$

PROOF. Note that (c) can be rewritten as

$$U = A\phi(U)A_0^{-1}.$$

This gives the first inequality. To deduce the second inequality, we proceed as in the proof of Theorem 16.2.1, except that we iterate $\lceil \log_q i \rceil$ times to get to the case $i = 0$ (rather than iterating $\lfloor \log_q i \rfloor$ times to get to the case $0 < i < p$). $\square$

THEOREM 16.3.3. *Let $N = \sum_{i=0}^\infty N_i t^i$, $U = \sum_{i=0}^\infty U_i t^i$, $A = \sum_{i=0}^\infty A_i t^i$ be $n \times n$ matrices over $K[\![t]\!]$ such that:*

(a) $|A|_1 < \infty$;
(b) $U_0 = I_n$ *and* $A_0$ *is invertible;*
(c) $U^{-1}NU + U^{-1}t\frac{d}{dt}(U) = N_0$;
(d) $NA + t\frac{d}{dt}(A) = qA\phi(N)$.

*Then $U^{-1}A\phi(U) = A_0$, and for every nonnegative integer $i$,*

$$|U_i| \leq (|A_0^{-1}||A|_1)^{\lceil \log_q i \rceil}.$$

PROOF. As noted in Remark 15.1.2, the commutation relation (d) implies that $N_0 A_0 = qA_0\phi(N_0)$, which forces $N_0$ to be nilpotent. Put $B = U^{-1}A\phi(U) = \sum_{i=0}^\infty B_i t^i$. Then $B_0 = A_0$, and $N_0 B + t\frac{d}{dt}(B) = qB\phi(N_0)$. Hence

$$N_0 B_i + iB_i = qB_i\phi(N_0) = B_i A_0^{-1} N_0 A_0,$$

or

$$N_0(B_i A_0^{-1}) + i(B_i A_0^{-1}) = (B_i A_0^{-1})N_0. \tag{16.3.3.1}$$

As in the proof of Proposition 16.1.1, the operator $X \mapsto N_0 X - XN_0 + iX$ on $n \times n$ matrices is invertible for $i \neq 0$, so (16.3.3.1) implies $B_i = 0$ for $i > 0$.

We conclude that indeed $U^{-1}A\sigma(U) = A_0$, so we may conclude by applying Lemma 16.3.2 to reduce to the case $i < q$, then applying Theorem 16.2.4. □

REMARK 16.3.4. By combining Theorem 16.2.4 with Theorem 16.2.1 (applying the latter for $i < q$), we can obtain the bound

$$|U_i| \leq |N|_1^{n-1} p^{(n-1)\lfloor \log_p i - (\log_p q)\lfloor \log_q i \rfloor \rfloor} (|A_0^{-1}||A|_1)^{\lfloor \log_q i \rfloor}.$$

REMARK 16.3.5. In applications to Picard-Fuchs modules, the difference between the bounds given by Theorem 16.2.4 and Theorem 16.3.3 can be quite significant. For instance, given a Picard-Fuchs module arising from a family of curves of genus $g$, the bound of Theorem 16.2.4 contains the factor $p^{(2g-1)\lfloor \log_p i \rfloor}$, but the bound of Theorem 16.3.3 replaces the factor of $2g - 1$ by 1. In general, it should be possible to use Theorem 16.3.3 (and perhaps also Theorem 16.3.6) to explain various instances in which a calculation of $n$ terms of a power series involves a precision loss of $p^{O(\log(n))}$, even though the accumulated factors of $p$ by which one divides throughout the calculation amount to $p^{O(n)}$. (A typical example of this is [**Ked03**, Lemma 3].)

We record also a sharper form of Theorem 16.3.3 for use in the discussion of logarithmic growth in the next section.

THEOREM 16.3.6. *Let $v$ be a column vector of length $n$ over $K[\![t]\!]$, let $A = \sum_{i=0}^\infty A_i t^i$ be an $n \times n$ matrix over $K[\![t]\!]$, and let $\lambda \in K$ be such that:*

(a) $|A|_1 < \infty$;
(b) $A_0$ *is invertible;*
(c) $A\sigma(v) = \lambda v$.

*Then*

$$\max\{|v_j| : 0 \leq j \leq i\} \leq |A|_1|\lambda^{-1}| \max\{|v_j| : 0 \leq j \leq i/q\}.$$

*Consequently, for every nonnegative integer $i$,*

$$|v_i| \leq |v_0|(|A|_1|\lambda^{-1}|)^{\lceil \log_q i \rceil}.$$

Proof. Rewrite (c) as $v = \lambda^{-1} A \sigma(v)$ and proceed as in Lemma 16.3.2. □

## 4. Logarithmic growth

Definition 16.4.1. For $\delta \geq 0$, let $K[\![t]\!]_\delta$ be the subset of $K[\![t]\!]$ consisting of those $f = \sum_{i=0}^{\infty} f_i t^i$ for which

$$|f|_\delta = \sup_i \left\{ \frac{|f_i|}{(i+1)^\delta} \right\} < \infty;$$

note that $K[\![t]\!]_\delta$ forms a Banach space under the norm $|\cdot|_\delta$. (The notation for $\delta = 0$ is consistent with our earlier usage.) However, $K[\![t]\!]_\delta$ is not a ring for $\delta > 0$; rather, we have

$$K[\![t]\!]_{\delta_1} \cdot K[\![t]\!]_{\delta_2} \subset K[\![t]\!]_{\delta_1+\delta_2}.$$

Also, $K[\![t]\!]_\delta$ is stable under $\frac{d}{dt}$, but antidifferentiation carries it into $K[\![t]\!]_{\delta+1}$. Put

$$K[\![t]\!]_{\delta+} = \bigcap_{\delta' > \delta} K[\![t]\!]_{\delta'}.$$

We also consider a logarithmic version:

$$K[\![t]\!][\log t]_\delta = \bigoplus_{i=0}^{\lfloor \delta \rfloor} K[\![t]\!]_{\delta-i} (\log t)^i.$$

For another useful characterization of $K[\![t]\!]_\delta$, see the exercises.

Definition 16.4.2. For $f \in K[\![t]\!][\log t]$, we say that $f$ has *order of log-growth $\delta$* if $f \in K[\![t]\!][\log t]_\delta$ but $f \notin K[\![t]\!][\log t]_{\delta'}$ for any $\delta' < \delta$. We say $f$ has *order of log-growth $\delta+$* if $f \notin K[\![t]\!][\log t]_\delta$ but $f \in K[\![t]\!][\log t]_{\delta'}$ for any $\delta' > \delta$. We have similar definitions for vectors or matrices over $K[\![t]\!][\log t]$, and for elements of $M \otimes K[\![t]\!][\log t]$ if $M$ is a finite free module over $K[\![t]\!]_0$ (by computing in terms of a basis, the choice of which will not affect the answer).

We then deduce the following from Theorem 16.2.4.

Proposition 16.4.3. *Let $M$ be a differential module of rank $n$ over $K[\![t]\!]_0$ for the operator $t\frac{d}{dt}$, which is nilpotent at the origin. Then $M \otimes K[\![t]\!]_{n-1}[\log t]$ is trivial.*

Corollary 16.4.4. *Let $M$ be a differential module of rank $n$ over $K[\![t]\!]_0$ for the operator $t\frac{d}{dt}$, which is nilpotent at the origin with index of nilpotency $e$. Then any element of $H^0(M \otimes K[\![t]\!][\log t])$ has order of log-growth at most $n - 1 + e$.*

Remark 16.4.5. In Corollary 16.4.4, it should be possible to reduce $n - 1 + e$ to $n$.

In the presence of a Frobenius structure, one obtains a much sharper bound.

Theorem 16.4.6. *Let $M$ be a differential module of rank $n$ over $K[\![t]\!]_0$ for the operator $t\frac{d}{dt}$, equipped with a Frobenius structure for a $q$-power Frobenius lift as in Remark 15.1.2. Then any element $v \in H^0(M \otimes K[\![t]\!][\log t])$ satisfying $\Phi(v) = \lambda v$ for some $\lambda \in K$ has order of log-growth at most $(-\log|\lambda| - s_0)/(\log q)$, where $s_0$ is the least generic Newton slope of $M$.*

Proof. By replacing the Frobenius lift by some power, we can reduce to the case where $s_0$ is a multiple of $-\log p$. We can then twist into the case $s_0 = 0$. By Proposition 13.5.8, we can choose a basis of $M$ such that the least generic Hodge slope of $M$ is also 0. Then the claim follows immediately from Theorem 16.3.6. □

Remark 16.4.7. Refining a conjecture of Dwork, Chiarellotto and Tsuzuki [**CT06**] have conjectured that if $M$ is indecomposable, then Theorem 16.4.6 is optimal. That is, in the notation of Theorem 16.4.6, $v$ should have order of log-growth exactly $(-\log|\lambda| - s_0)/(\log q))$; Chiarellotto and Tsuzuki have proven this for $\mathrm{rank}(M) \leq 2$ [**CT06**, Theorem 7.2]. It should be possible to extend their proof to all cases where $-\log|\lambda|$ is less than or equal to $s_1$ (the least Newton slope of $M$ greater than $s_0$, not counting multiplicity), but it is less clear what happens in general.

Remark 16.4.8. By contrast, if $M$ does not carry a Frobenius structure, then the order of log-growth of a horizontal section behaves much less predictably. For instance, it need not be rational, and it can have the form $\delta+$ instead of $\delta$ [**CT06**, §5.2].

## 5. Nonzero exponents

So far, we only have considered regular differential systems with all exponents equal to zero. Concerning nonzero exponents, we limit ourselves to two remarks.

Remark 16.5.1. Suppose the eigenvalues of $N_0$ are rational numbers with least common denominator dividing $m$. One can then apply Theorem 16.2.1 after making the substitution $t \mapsto t^m$, resulting in the bound

$$|U_i|\beta^i \leq p^{(n-1)\lfloor \log_p(im)\rfloor} \leq p^{(n-1)\lfloor \log_p m\rfloor} p^{(n-1)\lfloor \log_p i\rfloor}.$$

Note that as $i$ varies, the difference between the bound in this case and in the nilpotent case is only a constant multiplicative factor.

Remark 16.5.2. Suppose that the eigenvalues of $N_0$ all belong to $\mathbb{Z}_p$. (One might want to consider this remark instead of Remark 16.5.1 even if the eigenvalues are rational, in case one does not have an *a priori* bound on their denominators.) One can then prove an effective bound by imitating the proof of Theorem 16.2.1, but using shearing transformations to force the exponents to be multiples of $p$ before forming the Frobenius antecedent. However, the best known bound using this technique is worse than in Remark 16.5.1; it has the form $p^{(n^2+cn)\lfloor \log_p i\rfloor}$ for some constant $c$. See [**DGS94**, Theorem V.9.1] for more details.

## Notes

In the case of no singularities ($N_0 = 0$), the effective bound of Theorem 16.2.4 is due to Dwork and Robba [**DR80**], with a slightly stronger bound: one may replace $p^{(n-1)\lfloor \log_p i\rfloor}$ with the maximum of $|j_1 \cdots j_{n-1}|^{-1}$ over $j_1, \dots, j_{n-1} \in \mathbb{Z}$ with $1 \leq j_1 < \cdots < j_{n-1} \leq i$. See also [**DGS94**, Theorem IV.3.1].

The general case of Theorem 16.2.1 is due to Christol and Dwork [**CD91**], except that their bound is significantly weaker: it is roughly $p^{c(n-1)\lfloor \log_p i \rfloor}$ with $c = 2 + 1/(p-1)$. The discrepancy comes from the fact that the role of Proposition 5.2.11 is played in [**CD91**] by an effective version of the cyclic vector theorem, which does not give optimal bounds. As usual, use of cyclic vectors also introduces singularities which must then be removed, leading to some technical difficulties. See also [**DGS94**, Theorem V.2.1]. (The poor estimate in the case of exponents in $\mathbb{Z}_p$ does not appear to be due to use of cyclic vectors.)

In the case of no singularities, Proposition 16.4.3 was first proved by Dwork; it appears in [**Dwo73a**] and [**Dwo73b**]. (See also [**Chr83**].) The nilpotent case appears to be original; as noted above, the effective bounds in [**CD91**] are not strong enough to imply this. Theorems 16.3.3 and 16.3.6 are original, but they owe a great debt to the proof of [**CT06**, Theorem 7.2]; the main difference is that we prefer to argue in terms of matrices rather than cyclic vectors.

The theory of logarithmic growth in the $p$-adic setting (which may be viewed as loosely analogous to its archimedean counterpart, as in [**Del70**]) emerged from some close analysis made by Dwork [**Dwo73a, Dwo73b**] of the finer convergence behavior of solutions of certain $p$-adic differential equations. The subject languished until the recent work of Chiarellotto and Tsuzuki [**CT06**]; inspired by this, André [**And07**] proved a conjecture of Dwork [**Dwo73b**, Conjecture 2] analogizing the specialization property of Newton polygons (Theorem 14.3.2) to logarithmic growth.

### Exercises

(1) Prove that for $\delta \geq 0$,

$$K[\![t]\!]_\delta = \{f \in K[\![t]\!] : \limsup_{\rho \to 1^-} \frac{|f|_\rho}{(-\log \rho)^\delta} < \infty.\}$$

(Hint: the inequality

$$\sup_i \{(i+1)^\delta \rho^i\} \leq \rho^{-1} \left(\frac{\delta}{e}\right)^\delta (-\log \rho)^{-\delta}$$

may be helpful.)