CHAPTER 12

p-adic exponents

In this chapter, we discuss (without full proofs) what happens when one tries to analyze p-adic differential modules on annuli for which the intrinsic generic radius of convergence is equal to 1 everywhere; this is precisely the case where the techniques of the previous chapters fail to deliver any information. It turns out that there is a notion of p-adic exponents in this setting, but one must avoid exponents which are closely approximated by integers without being integers themselves (p-adic Liouville numbers). This can already be seen by considering p-adic differential modules on discs with one regular singularity, so we do that first.

1. p-adic Liouville numbers

DEFINITION 12.1.1. For $\lambda \in K$, the *type* of λ, denoted $\mathrm{type}(\lambda)$, is the radius of convergence of the p-adic power series

$$\sum_{m=0, m\neq\lambda}^{\infty} \frac{x^m}{\lambda - m}.$$

This cannot exceed 1, as there are infinitely many m for which $|\lambda - m| = 1$ (namely those not congruent to λ modulo p). Moreover, if $\lambda \notin \mathbb{Z}_p$, then $|\lambda - m|$ is bounded below, so $\mathrm{type}(\lambda) = 1$. We will thus mostly worry about $\lambda \in \mathbb{Z}_p$.

DEFINITION 12.1.2. We say that λ is a *p-adic Liouville number* if either λ or $-\lambda$ has type less than 1, and a *p-adic non-Liouville number* otherwise. The reference to both λ and $-\lambda$ is not superfluous, as they may have different types (exercise).

The following alternate characterization of type may be helpful.

DEFINITION 12.1.3. For $\lambda \in \mathbb{Z}_p$, let $\lambda^{(m)}$ be the unique integer in $\{0, \dots, p^m - 1\}$ congruent to λ modulo p^m.

PROPOSITION 12.1.4. *For $\lambda \in \mathbb{Z}_p$ not a nonnegative integer,*

$$-\frac{1}{\log_p \mathrm{type}(\lambda)} = \liminf_{m\to\infty} \frac{\lambda^{(m)}}{m}. \tag{12.1.4.1}$$

In particular, λ has type 1 if and only if $\lambda^{(m)}/m \to \infty$ as $m \to \infty$.

PROOF. It suffices to check that for $0 < \eta < 1$, we have

$$\limsup_{m\to\infty} (m + \lambda^{(m)} \log_p \eta) = -\infty \tag{12.1.4.2}$$

when $\eta < \mathrm{type}(\alpha)$ and

$$\limsup_{m\to\infty} (m + \lambda^{(m)} \log_p \eta) = +\infty \tag{12.1.4.3}$$

when $\eta > \operatorname{type}(\alpha)$. Namely, (12.1.4.2) implies $m + \lambda^{(m)} \log_p \eta \leq 0$ for all large m, so $\liminf_{m\to\infty} \frac{\lambda^{(m)}}{m} \geq -1/(\log_p \eta)$, whereas (12.1.4.3) implies $m + \lambda^{(m)} \log_p \eta \geq 0$ for infinitely many m, so $\liminf_{m\to\infty} \frac{\lambda^{(m)}}{m} \leq -1/(\log_p \eta)$.

Suppose first that $\operatorname{type}(\alpha) > \eta > 0$; then as $s \to \infty$, $\eta^s/|\lambda - s| \to 0$ or equivalently $v_p(\lambda - s) + s \log_p \eta \to -\infty$. (Here v_p denotes the renormalized valuation with $v(p) = 1$.) Since λ is not a nonnegative integer, we have $\lambda^{(m)} \to \infty$ as $m \to \infty$, so

$$v_p(\lambda - \lambda^{(m)}) + \lambda^{(m)} \log_p \eta \to -\infty.$$

The left side does not increase if we replace $v_p(\lambda - \lambda^{(m)})$ by m, so we may deduce (12.1.4.2).

Suppose next that $\operatorname{type}(\alpha) < \eta < 1$; then we may choose a sequence s_j such that as $j \to \infty$, $v_p(\lambda - s_j) + s_j \log_p \eta \to +\infty$. Put $m_j = v_p(\lambda - s_j)$, so that $s_j \geq \lambda^{(m_j)}$. Then

$$m_j + \lambda^{(m_j)} \log_p \eta \to +\infty,$$

yielding (12.1.4.3). □

The alternate characterization is convenient for such verifications as the fact that rational numbers are non-Liouville (exercise), or this stronger result [**DGS94**, Proposition VI.1.1], whose proof we omit.

PROPOSITION 12.1.5. *Any element of $\mathbb{Z}_p$ algebraic over $\mathbb{Q}$ is non-Liouville.*

We will encounter the p-adic Liouville property in yet another apparently different form. (See exercises for an alternate proof of this lemma.)

LEMMA 12.1.6. *For λ not a nonnegative integer, we have an equality of formal power series*

$$\sum_{m=0}^{\infty} \frac{x^m}{\lambda(1-\lambda)(2-\lambda)\cdots(m-\lambda)} = e^x \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} \frac{1}{\lambda - m}.$$

PROOF. The coefficient of x^m on the right side is a sum of the form $\sum_{i=0}^{m} c_i/(i - \lambda)$ for some $c_i \in \mathbb{Q}$. It is thus a rational function of λ of the form $P(\lambda)/(\lambda(1-\lambda)\cdots(m-\lambda))$, where P has coefficients in $\mathbb{Q}$ and degree at most m. To check that in fact $P(\lambda) = 1$ identically, we need only check this for $\lambda = 0, \dots, m$.

In other words, to check the original identity, it suffices to check after multiplying both sides by $\lambda - i$ and evaluating at $\lambda = i$, for each nonnegative integer i. On the left side, we obtain

$$\sum_{m=i}^{\infty} \frac{-x^m}{(-1)^{i-1} i! (m-i)!}.$$

On the right side, we obtain

$$e^x \frac{(-x)^i}{i!},$$

which is the same thing. □

COROLLARY 12.1.7. *If $\lambda \in K$ is not a nonnegative integer, and* $\operatorname{type}(\lambda) = 1$, *then the series*

$$\sum_{m=0}^{\infty} \frac{x^m}{\lambda(1-\lambda)(2-\lambda)\cdots(m-\lambda)}$$

has radius of convergence $p^{-1/(p-1)}$.

2. p-adic regular singularities

We now consider a p-adic analogue of Theorem 6.3.5. Unlike its archimedean analogue, it requires a hypothesis on exponents beyond simply being weakly prepared (which simply meant that no two eigenvalues of the constant matrix differ by a nonzero integer).

DEFINITION 12.2.1. We say that a finite set is *p-adic non-Liouville* if its elements are p-adic non-Liouville number. We say the set has *p-adic non-Liouville differences* if the difference between any two elements of the set is a p-adic non-Liouville number.

THEOREM 12.2.2 (p-adic Fuchs theorem). *For $\beta > 0$, let M be a finite differential module on $K\langle t/\beta\rangle$ for the derivation $d = t\frac{d}{dt}$. Let $N = \sum_{i=0}^{\infty} N_i t^i$ be the action of D on some basis. Assume that N_0 has eigenvalues which are weakly prepared and have p-adic non-Liouville differences. Then there exists $\gamma > 0$ such that the fundamental solution matrix for N has entries in $K\langle t/\gamma\rangle$ (as does its inverse).*

PROOF. We proceed as in Proposition 16.1.1. Recall (6.3.4.1):

$$N_0 U_i - U_i N_0 + i U_i = -\sum_{j=0}^{i-1} N_j U_{i-j}.$$

Because N_0 has weakly prepared eigenvalues, U is uniquely determined. There is thus no harm in enlarging K to ensure that the eigenvalues $\lambda_1, \dots, \lambda_n$ of N_0 belong to K. Then the map $X \mapsto N_0 X - X N_0 + i$ has eigenvalues $\lambda_g - \lambda_h + i$ for $g, h \in \{1, \dots, n\}$. If e is the maximum number of pairwise equal eigenvalues, we obtain the bound

$$|U_i|\beta^i \leq \max_{g,h}\{|\lambda_g - \lambda_h + i|^{-2e+1}\}|N|_\beta \max_{j<i}\{|U_j|\beta^j\}.$$

Thus to conclude the theorem, it suffices to verify that for each $h, j \in \{1, \dots, n\}$, the number $\lambda = \lambda_g - \lambda_h$ has the property that

$$\prod_{i=1}^{m} \max\{1, |\lambda - i|^{-1}\}$$

grows at worst exponentially.

If $\lambda \notin \mathbb{Z}_p$, then $|\lambda - i|^{-1}$ is bounded above and the claim is verified. Otherwise, Corollary 12.1.7 and the hypothesis that λ is a p-adic non-Liouville number give the desired estimate. $\square$

By a slight modification of the argument (which we omit), one may obtain the following result of Clark [**Cla66**, Theorem 3].

THEOREM 12.2.3 (Clark). *Let M be a finite differential module over $K\langle t/\beta\rangle$ for the derivation $t\frac{d}{dt}$, with a regular singularity at 0 whose exponents are p-adic non-Liouville numbers. Then for any $x \in M$ and $y \in M \otimes K[\![t]\!]$ such that $Dy = x$, we have $Dy \in M \otimes K\langle t/\rho\rangle$ for some $\rho > 0$.*

The p-adic non-Liouville hypothesis in Theorem 12.2.2 turns out not to be superfluous, as demonstated by the following example of Monsky.

Example 12.2.4. Consider the rank 2 differential module over $K\langle t\rangle$ for the derivation $t\frac{d}{dt}$ associated to the differential polynomial $p(1-t)T^2 - tT - a$, where $a \in \mathbb{Z}_p$ is constructed so that

$$\text{type}(a) = 1, \qquad \text{type}(-a) < 1. \tag{12.2.4.1}$$

(The existence of such a is left as an exercise, or see [**DR77**, §7.20].) It can then be shown that the conclusion of Theorem 12.2.2 fails for the basis $1, T$ of M, that is, the fundamental solution matrix does not converge in any disc. (The eigenvalues of N_0 are $0, a$, so the hypothesis of non-Liouville differences is violated by this example.) See [**DR77**, §7] or [**DGS94**, §IV.8] for further discussion.

3. The Robba condition

We are interested in the question: given a finite differential module on an annulus for the derivation $t\frac{d}{dt}$, under what circumstances is it necessarily isomorphic to a differential module which can be defined over a disc?

In order to answer this question, we must identify properties of a differential module on a disc which betray information about the exponents, but which are defined in terms of information away from the center of the disc.

Definition 12.3.1. Let M be a finite differential module on the disc/annulus $|t| \in I$, for I an interval. We say that M satsifies the *Robba condition* if $IR(M \otimes F_\rho) = 1$ for all nonzero $\rho \in I$.

Proposition 12.3.2. *Let M be a finite differential module on the open disc of radius β for the derivation $t\frac{d}{dt}$, satisfying the Robba condition in some annulus. Then the exponents of the action of D on M/tM belong to $\mathbb{Z}_p$.*

Proof. Let $N = \sum_{i=0}^{\infty} N_i t^i$ be the matrix via which D acts on some basis of M. Suppose N_0 has an eigenvalue $\lambda \notin \mathbb{Z}_p$; there is no harm in enlarging K to force $\lambda \in K$. Choose $v \in M$ such that the image of v in M/tM is a nonzero eigenvector of N_0 of eigenvalue λ. Let D' be the derivation corresponding to $\frac{d}{dt}$ instead of $t\frac{d}{dt}$. Then with notation as in Example 8.2.5, we have for any $\rho < \beta$,

$$\liminf_{s\to\infty} |(D')^s v|^{1/s} > |D'|_{\mathrm{sp},V_\lambda,\rho} > p^{-1/(p-1)}\rho,$$

so $IR(M \otimes F_\rho) < 1$. □

We will establish a partial converse to Proposition 12.3.2 later (Theorem 12.7.1). In the interim, we mention the following easy result.

Proposition 12.3.3. *Let M be a finite differential module on the open disc of radius β for the derivation $t\frac{d}{dt}$, such that the action of D on some basis of M is given by a matrix N_0 over K. Then M satisfies the Robba condition if and only if N_0 has eigenvalues in $\mathbb{Z}_p$.*

Proof. Exercise, or see [**DGS94**, Corollary IV.7.6]. □

4. Abstract p-adic exponents

We now consider the question: given a finite differential module on an annulus for the derivation $t\frac{d}{dt}$ satisfying the Robba condition, if it is isomorphic to a differential module over a disc, how do we read off the exponents of that module by looking only at the original annulus?

The answer to this question is complicated by the fact that the exponents are only well-defined as elements of the quotient $\mathbb{Z}_p/\mathbb{Z}$. This means we cannot hope to identify them using purely p-adic considerations; in fact, we must use *archimedean* considerations to identify them. Here are those considerations.

Definition 12.4.1. We will say that two elements $A, B \in \mathbb{Z}_p^n$ are *equivalent* if there exists a permutation σ of $\{1, \dots, n\}$ such that $A_i - B_{\sigma(i)} \in \mathbb{Z}$ for $i = 1, \dots, n$. This is evidently an equivalence relation.

Definition 12.4.2. We say that $A, B \in \mathbb{Z}_p^n$ are *weakly equivalent* if there exists a constant $c > 0$, a sequence $\sigma_1, \sigma_2, \dots$ of permutations of $\{1, \dots, n\}$, and signs $\epsilon_{i,m} \in \{\pm 1\}$ such that

$$(\epsilon_{i,m}(A_i - B_{\sigma_m(i)}))^{(m)} \leq cm \qquad (i = 1, \dots, n; m = 1, 2, \dots).$$

In other words, the distance from $A_i - B_{\sigma_m(i)}$ to the nearest multiple of p^m is at most cm. Again, this is clearly an equivalence relation, and equivalence implies weak equivalence.

Lemma 12.4.3. *If $A, B \in \mathbb{Z}_p$ (regarded as 1-tuples) are weakly equivalent, then they are equivalent.*

Proof. For some $c > 0$, we have

$$|\epsilon_{1,m+1}(\epsilon_{1,m+1}(A - B))^{(m+1)} - \epsilon_{1,m}(\epsilon_{1,m}(A - B))^{(m)}| \leq 2cm + c,$$

and the left side is an integer divisible by p^m. For m large enough, we have $p^m > 2cm + c$ and so

$$\epsilon_{1,m+1}(\epsilon_{1,m+1}(A - B))^{(m+1)} = \epsilon_{1,m}(\epsilon_{1,m}(A - B))^{(m)}.$$

Hence for m enough, $\epsilon_{1,m}$ is constant and $\epsilon_{1,m}(A - B)$ is a nonnegative integer. $\square$

Corollary 12.4.4. *Suppose $A \in \mathbb{Z}_p^n$ is weakly equivalent to hA for some positive integer h. Then $A \in (\mathbb{Z}_p \cap \mathbb{Q})^n$.*

Proof. We are given that for some $c > 0$, some permutations σ_m, and some signs $\epsilon_{i,m}$,

$$(\epsilon_{i,m}(A_i - hA_{\sigma_m(i)}))^{(m)} \leq cm.$$

The order of σ_m divides $n!$, so we have

$$(\pm(A_i - h^{n!}A_i))^{(m)} \leq n!cm$$

for some choice of sign (depending on i, m). That is, for each i, the 1-tuple consisting of $(h^{n!}-1)A_i$ is weakly equivalent to zero. By Lemma 12.4.3, $(h^{n!}-1)A_i \in \mathbb{Z}$, so $A_i \in \mathbb{Z}_p \cap \mathbb{Q}$. $\square$

Proposition 12.4.5. *Suppose that $A, B \in \mathbb{Z}_p^n$ are weakly equivalent and that B has p-adic non-Liouville differences. Then A and B are equivalent.*

PROOF. There is no harm in replacing A by an equivalent tuple in which $B_i - B_j \in \mathbb{Z}$ if and only if $B_i = B_j$.

For some c and σ_m, we have for all m,

$$(\pm(A_i - B_{\sigma_m(i)}))^{(m)} \leq cm$$
$$(\pm(A_i - B_{\sigma_{m+1}(i)}))^{(m+1)} \leq c(m+1)$$

and so

$$(\pm(B_{\sigma_m(i)} - B_{\sigma_{m+1}(i)}))^{(m)} \leq 2cm + c.$$

By hypothesis, the difference $B_{\sigma_m(i)} - B_{\sigma_{m+1}(i)}$ is either zero or a p-adic non-Liouville number which is not an integer; for m large, the previous inequality is inconsistent with the second option, so $B_{\sigma_m(i)} = B_{\sigma_{m+1}(i)}$. That is, for m large we have $\sigma_m = \sigma$ for some fixed σ, so

$$(\pm(A_i - B_{\sigma(i)}))^{(m)} \leq cm \qquad (m = 1, 2, \dots).$$

By Lemma 12.4.3, $A_i - B_{\sigma(i)} \in \mathbb{Z}$, so A and B are equivalent. $\square$

5. Exponents for annuli

DEFINITION 12.5.1. Let M be a finite differential module of rank n over $K\langle \alpha/t, t/\beta \rangle$ satisfying the Robba condition, and fix a basis $e_1, \dots, e_n$ of M. An *exponent* for M is an element $A \in \mathbb{Z}_p^n$ for which there exist a sequence $\{S_m\}_{m=1}^\infty$ of $n \times n$ matrices over $K\langle \alpha/t, t/\beta \rangle$ satisfying the following conditions.

(a) For $j = 1, \dots, n$, under the action of ζ_{p^m} on M via Taylor series (which converge because of the Robba condition), the vector $v_{m,j} = \sum_i (S_m)_{ij} e_i$ is carried to $\zeta_{p^m}^{A_j} v_{m,j}$.
(b) For some k, we have $|S_m|_\rho \leq p^{mk}$ for all m and all $\rho \in [\alpha, \beta]$.
(c) Writing $S_m = \sum_{h \in \mathbb{Z}} S_{m,h} t^h$, we have $|S_{m,0}|_\rho \geq 1$ for all $\rho \in [\alpha, \beta]$.

Note that the property of being an exponent does not depend on the choice of the basis (although the choice of the matrices S_m does).

PROPOSITION 12.5.2. *Let M be a finite differential module of rank n over $K\langle \alpha/t, t/\beta \rangle$ satisfying the Robba condition.*

(a) *There exists an exponent for M.*
(b) *Any two exponents for M are weakly equivalent. In particular, if M admits an exponent with non-Liouville differences, then (by Lemma 12.4.3) any other exponent for M is strongly equivalent to it.*

PROOF. For (a), see [**Dwo97**, Lemma 3.1, Corollary 3.3]. For (b), see [**Dwo97**, Theorem 4.4]. $\square$

REMARK 12.5.3. If M is a differential module of rank n over $K\langle t/\beta \rangle$ for the derivation $t\frac{d}{dt}$, such that the eigenvalues of the action of D on M/tM are in $\mathbb{Z}_p$, then it is easy to check (using shearing transformations) that these eigenvalues form an exponent for $M \otimes K\langle \alpha/t, t/\beta \rangle$ for any $\alpha \in (0, \beta)$.

The following is straightforward to verify.

LEMMA 12.5.4. *Let M be a finite differential module of rank n over $K\langle \alpha/t, t/\beta \rangle$ satisfying the Robba condition, and let $\phi : K\langle \alpha/t, t/\beta \rangle \to K\langle \alpha^{1/q}/t, t/\beta^{1/q} \rangle$ be the substitution $t \mapsto t^q$. If A is an exponent of M, then qA is an exponent of $\phi^* M$.*

Corollary 12.5.5. *Let M be a finite differential module on an open annulus with outer radius 1, admitting a Frobenius structure. Then any exponent for M consists of rational numbers.*

Proof. This holds by Lemma 12.5.4 and Corollary 12.4.4. $\square$

6. The p-adic Fuchs theorem for annuli

Having sufficiently well understood the definition of exponents for a differential module on an open annulus, one then obtains the following theorem. We omit its proof; see notes for further discussion.

Theorem 12.6.1 (Christol-Mebkhout). *Let M be a finite differential module on an open annulus for the derivation $t\frac{d}{dt}$ satisfying the Robba condition, admitting an exponent with non-Liouville differences. Then M is isomorphic to a differential module in which D acts on some basis via a matrix N_0 with coefficients in K, whose eigenvalues represent the exponents of M (and hence are in $\mathbb{Z}_p$). Consequently, M admits a canonical decomposition*

$$M = \bigoplus_{\alpha \in \mathbb{Z}_p/\mathbb{Z}} M_\alpha$$

in which each M_α has exponents identically equal to α.

Remark 12.6.2. The exponent differences condition is difficult to verify in general because of the indirect nature of the definition of exponents. However, if M is a finite differential admits a Frobenius structure, then Corollary 12.5.5 implies that the exponents are rational. This leads to a quasiunipotence result (Theorem 18.3.1) which can be used to establish the p-adic local monodromy theorem (Theorem 18.1.8).

7. Transfer to a regular singularity

As an application of Theorem 12.6.1, we obtain a transfer theorem in the presence of a regular singularity, in the spirit of Theorem 8.5.1 and Theorem 8.5.4 but with a somewhat weaker estimate.

Theorem 12.7.1. *Let M be a finite differential module of rank n over $K[\![t]\!]_0$ for the derivation $t\frac{d}{dt}$, with a regular singularity at $t = 0$ whose exponents are in $\mathbb{Z}_p$ and have non-Liouville differences. Then the fundamental solution matrix of M converges in the open disc of radius $R(M \otimes F_1)^n$. In particular, if M has generic radius of convergence 1, then the fundamental solution matrix of M converges in the open unit disc.*

Proof. By Theorem 12.2.2, the fundamental solution matrix of M converges in a disc of positive radius. From this and Proposition 12.3.3, it follows that $R(M \otimes F_\rho) = \rho$ for $\rho \in (0, 1)$ sufficiently small.

Let λ be the supremum of $\rho \in (0, 1)$ for which $R(M \otimes F_\rho) = \rho$. Note that the function $f_1(r) = -\log R(M \otimes F_{e^{-r}})$ is convex by Theorem 10.3.2, is equal to r for r sufficiently large by the previous paragraph, and is also equal to r for $r = -\log \lambda$ by continuity. Consequently, $f_1(r) = r$ for all $r \geq -\log \lambda$.

Choose $\alpha, \beta \in (0, \lambda)$ with $\alpha < \beta$, such that the fundamental solution matrix of M converges in the open disc of radius β. By Theorem 12.6.1, it also converges in the open annulus of inner radius α and outer radius 1. By patching, we deduce that the fundamental solution matrix converges in the open disc of radius λ.

To conclude, it suffices to give a lower bound for λ. By Theorem 10.3.2, for $r \in [0, -\log \lambda]$, the function f_1 is continuous and piecewise affine, with slopes belonging to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}$. Since the slope for $r > -\log \lambda$ is equal to 1, the slopes for $r \leq -\log \lambda$ cannot exceed 1; moreover, there cannot be a slope equal to 1 in this range, as otherwise it would occur as the left slope at $r = -\log \lambda$, so there would exist $\rho > \lambda$ for which $R(M \otimes F_\rho) = \rho$, contrary to how λ was defined. Consequently, f_1 has all slopes less than or equal to $(n-1)/n$ for $r \in [0, -\log \lambda]$, yielding

$$-\log \lambda = f_1(-\log \lambda) \leq f_1(0) + \frac{n-1}{n}(-\log \lambda).$$

From this we deduce $\lambda \geq R(M \otimes F_1)^n$, as desired. □

Remark 12.7.2. We do not have in mind an example where one does not get convergence on the open disc of radius $R(M \otimes F_1)$.

Notes

The definition of a p-adic Liouville number was introduced by Clark [**Cla66**]; our presentation follows [**DGS94**, §VI.1].

The cited theorem of Clark [**Cla66**, Theorem 3] is actually somewhat stronger than Theorem 12.2.3, as it allows differential operators of possibly infinite order.

Proposition 12.3.2 is originally due to Christol; compare [**DGS94**, Proposition IV.7.7].

The theory of exponents for differential modules on a p-adic annulus satisfying the Robba condition was originally developed by Christol and Mebkhout [**CM97**, §4–5]; in particular, Theorem 12.6.1 appears therein as [**CM97**, Théorème 6.2–4]. A somewhat more streamlined development was later given by Dwork [**Dwo97**], in which Theorem 12.6.1 appears as [**Dwo97**, Theorem 7.1]. (Dwork coyly notes that he did not verify the equivalence between the two constructions; we do not recommend losing any sleep over this.) A useful expository article on the topic is that of Loeser [**Loe96**].

A somewhat more elementary treatment of Theorem 12.7.1 than the one given here is given in [**DGS94**, §6]; it does not rely on the p-adic Fuchs theorem for annuli. However, it gives a weaker result: it only establishes convergence of the fundamental solution matrix in the open disc of radius $R(M \otimes F_1)^{n^2}$. A similar treatment is [**Chr83**, Théorème 6.4.7].

Exercises

(1) Prove that rational numbers are p-adic non-Liouville numbers.
(2) Give another proof of Lemma 12.1.6 (as in [**DGS94**, Lemma VI.1.2]) by first verifying that both sides of the desired equation have the same coefficients of x^0 and x^1, and are killed by the second-order differential operator $\frac{d}{dx}(\frac{d}{dx} - \lambda - x)$.
(3) Show that Theorem 12.2.2 can be deduced from Theorem 12.2.3. (Hint: show that if $H^0(M) \neq 0$, then 0 must occur as an eigenvalue of N_0.)
(4) Prove that there exists $a \in \mathbb{Z}_p$ satisfying (12.2.4.1).
(5) Prove Proposition 12.3.3.