p-adic differential equations
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Frobenius structures on differential modules

In this unit, we construct Frobenius structures for differential modules on discs and annuli. We then explain how they arise for differential modules coming from geometry (Picard-Fuchs modules).

1 A few more rings

It will be useful to hybridize some notations for rings that I have already introduced. Recall that

$$K\langle \alpha/t, t/\beta \rangle = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i : c_i \in K, \lim_{i \to -\infty} |c_i| \alpha^i = 0, \lim_{i \to +\infty} |c_i| \beta^i = 0 \right\}.$$

We will also need

$$K[\![t]\!]_0 = \left\{ \sum_{i=0}^{\infty} c_i t^i : c_i \in K, \sup_i \{|c_i|\} < \infty \right\}$$
$$K\{\{t\}\} = \left\{ \sum_{i=0}^{\infty} c_i t^i : c_i \in K, \lim_{i \to \infty} |c_i| \rho^i = 0 \quad (\rho \in (0,1)) \right\}.$$

We will allow the following hybrids:

$$K\langle \alpha/t, t]\!]_0 = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i : c_i \in K, \lim_{i \to -\infty} |c_i| \alpha^i = 0, \sup_i \{|c_i|\} < \infty \right\}$$
$$K\langle \alpha/t, t\}\} = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i : c_i \in K, \lim_{i \to -\infty} |c_i| \alpha^i = 0, \lim_{i \to +\infty} |c_i| \rho^i = 0 \quad (\rho \in (0,1)) \right\}.$$

2 Frobenius structures

Let q be a power of p. Let $\phi_K : K \to K$ be any isometric endomorphism of K. Let R be one of the following rings:

- $K\langle t \rangle$, $K[\![t]\!]_0$, or $K\{\{t\}\}$;
- the union of $K\langle \alpha/t, t \rangle$, $K\langle \alpha/t, t]\!]_0$, or $K\langle \alpha/t, t\}\}$ over all $\alpha \in (0,1)$;
- F_1, the completion of $K(t)$ for the 1-Gauss norm;
- $\mathcal{E}$, the completion of $K[\![t]\!]_0[t^{-1}]$ for the 1-Gauss norm.

By a *q-power Frobenius lift* on R, we will mean a map $\phi : R \to R$ of the form

$$\sum_i c_i t^i \mapsto \sum_i \phi_K(c_i) u^i,$$

where $\phi_K : K \to K$ is an isometry, and $u \in R$ satisfies $|u - t^q|_1 < 1$. (Note that if $R = \cup_{\alpha>0} K\langle \alpha/t, t\}\}$, then the norm $|\cdot|_1$ is only defined on the subring $\cup_{\alpha>0} K\langle \alpha/t, t\rrbracket_0$.) We do not require ϕ_K itself to lift the q-power Frobenius on κ_K; if it does, we say that ϕ is an *absolute Frobenius lift*. (One could work somewhat more generally, allowing ϕ_K to carry K isometrically into $K\llbracket t \rrbracket_0$ in such a way that its composition with reduction modulo t is again an isometry.)

For M a finite free differential module over R, a *Frobenius structure* on M with respect to a Frobenius lift ϕ on R is an isomorphism $\Phi : \phi^* M \cong M$ of differential modules. In more explicit terms, we must equip M with the structure of a dualizable difference module over (R, ϕ), such that

$$D(\Phi(m)) = \frac{d\phi(t)}{dt} \Phi(D(v)) \qquad (m \in M).$$

In even more explicit terms, if A, N are the matrices via which Φ, D act on some bases, they must satisfy

$$NA + \frac{dA}{dt} = \frac{d\phi(t)}{dt} A\phi(N).$$

It is not easy to directly construct Frobenius structures except in a few simple examples. However, they can be shown to exist in many cases by more abstract methods; see below.

3 Frobenius structures and generic radius of convergence

In this section, let ϕ be a Frobenius lift on $\cup_{\alpha>0} K\langle \alpha/t, t\rrbracket_0$.

Lemma 1. *There exists $\epsilon \in (0,1)$ depending on ϕ, such that for $\beta, \gamma \in [\epsilon, 1)$ with $\beta \leq \gamma$, ϕ carries $K\langle \beta/t, t/\gamma\rangle$ to $K\langle \beta^{1/q}/t, t/\gamma^{1/q}\rangle$, and*

$$|f|_\beta = |\phi(f)|_{\beta^{1/q}}.$$

We can thus also talk about Frobenius structures with respect to ϕ on finite differential modules on the half-open annulus with closed inner radius α and open outer radius 1, whether or not they are not represented by finite free modules over $K\langle \alpha/t, t\}\}_0$. One of Dwork's early discoveries is that the presence of a Frobenius structure in this case forces solvability at the boundary.

Proposition 2. *Let M be a finite differential module on the half-open annulus with closed inner radius α and inner outer radius 1, equipped with a Frobenius structure. Then*

$$\lim_{\rho \to 1^-} IR(M \otimes F_\rho) = 1,$$

that is, M is solvable at the outer boundary. More precisely, for $\rho \in (0,1)$ sufficiently close to 1,

$$IR(M \otimes F_{\rho^{1/q}}) \geq IR(M \otimes F_\rho)^{1/q}.$$

Proof. By imitating the proof from the unit on Frobenius antecedents and descendants, we can show that for $\rho \in (0,1)$ sufficiently close to 1,

$$IR(M \otimes F_{\rho^{1/q}}) \geq \min\{IR(M \otimes F_\rho)^{1/q}, qIR(M \otimes F_\rho)\}.$$

The function $f(s) = \min\{s^{1/q}, qs\}$ on $(0,1]$ is strictly increasing, and any sequence of the form $s, f(s), f(f(s)), \dots$ converges to 1. This proves the first claim; for the second claim, note that once s is sufficiently close to 1, $f(s) = s^{1/q}$. □

The following corollary is sometimes called "Dwork's trick".

Corollary 3 (Dwork)**.** *Let M be a finite differential module on the open unit disc, equipped with a Frobenius structure. Then M admits a basis of horizontal sections.*

Proof. By Proposition 2, for each $\lambda < 1$, there exists $\rho \in (\lambda, 1)$ such that $R(M \otimes F_\rho) > \lambda$. By Dwork's transfer theorem, $M \otimes K\langle t/\lambda \rangle$ admits a basis of horizontal sections. Taking λ arbitrarily close to 1 yields the claim. □

4 Independence from the Frobenius lift

Another key property of Frobenius structures is that the exact shape of the Frobenius lift is immaterial.

Proposition 4. *Let ϕ_1, ϕ_2 be two Frobenius lifts on R. Let M be a finite free differential module over R equipped with a Frobenius structure for ϕ_1. Then there is a functorial way to equip M with a Frobenius structure for ϕ_2.*

Proof. The Frobenius structure for ϕ_2 is defined by

$$\Phi_2(m) = \sum_{i=0}^{\infty} \frac{(\phi_2(t) - \phi_1(t))^i}{i!} \Phi_1 \left(\frac{d^i}{dt^i}(m) \right),$$

where the series converges under $|\cdot|_\rho$ for $\rho \in (0,1)$ sufficiently close to 1, and also under $|\cdot|_1$ if $|\cdot|_1$ is defined on all of R. □

Corollary 5. *Let ϕ_1, ϕ_2 be two Frobenius lifts on R. Then there is a canonical equivalence between the categories of finite free differential modules over R equipped with Frobenius structure with respect to ϕ_i for $i = 1, 2$; this equivalence is the identity functor on the underlying difference modules.*

5 Picard-Fuchs modules

The reason for introducing Frobenius structures is that differential equations "arising from geometry" carry such structures. Here is what I mean by this.

Let t be a coordinate on $\mathbb{P}^1_K$. Let $f : X \to \mathbb{P}^1_K$ be a proper, flat, generically smooth morphism of algebraic varieties. Let $S \subset \mathbb{P}^1_K$ be a zero-dimensional subscheme containing ∞ (for convenience) and all points over which f is not smooth. The *Picard-Fuchs module* on $\mathbb{P}^1_K \setminus S$ associated to f is a finite locally free differential module M over $R = \Gamma(\mathbb{P}^1_K \setminus S, \mathcal{O})$ with respect to the derivation $\frac{d}{dt}$; it also has regular singularities at each point of S.

Although the classical construction of the Picard-Fuchs module is analytic (it involves viewing f as an analytically locally trivial fibration and integrating differentials against moving homology classes), there is an algebraic construction due to Katz and Oda [KO68], involving a Leray spectral sequence for the algebraic de Rham cohomology of the total space.

As originally noticed by Dwork by explicitly calculating some examples, Picard-Fuchs modules often carry Frobenius structures. A systematic explanation of this is given by p-adic cohomology; here is an explicit statement.

Theorem 6. *With notation as above, suppose that f extends to a proper morphism $\mathfrak{X} \to \mathbb{P}^1_{\mathfrak{o}_K}$ such that the intersection of $\mathbb{P}^1_k$ with the nonsmooth locus is contained in the intersection of $\mathbb{P}^1_k$ with the Zariski closure of S (i.e., the morphism is smooth over all points of $\mathbb{P}^1_k$ which are not the reductions of points in S). Let V_i be the i-th Picard-Fuchs module for f, and let $\phi : \mathbb{P}^1_{\mathfrak{o}_K} \to \mathbb{P}^1_{\mathfrak{o}_K}$ be a Frobenius lift (e.g., $t \mapsto t^p$) that acts on $\mathfrak{o}_K$ as a lift of the absolute Frobenius. Then for some $\alpha \in (0,1)$, there exists an isomorphism $\phi^* V_i \cong V_i$ over a ring R which is the Fréchet completion of $\Gamma(\mathbb{P}^1_K \setminus S, \mathcal{O})$ for (for $\rho \in [\alpha, 1)$) the ρ^{-1}-Gauss norm and the Gauss norms $|t - \lambda| = \rho$ for $\lambda \in S$.*

Geometrically, the Frobenius structure is defined on the complement in $\mathbb{P}^1_K$ of a union of discs around the points of S, each of radius less than 1 (where a disc of radius less than 1 around ∞ corresponds to the complement of a disc of radius greater than 1 around 0). In particular, by working in a unit disc not containing any points of S, we obtain a differential module with Frobenius structure over $K[\![t]\!]_0$. In a unit disc containing one or more points of S, we only obtain a differential module with Frobenius structure over $\cup_{\alpha>0} K\langle \alpha/t, t]\!]_0$. (If the disc contains exactly one point of S and the exponents at that point are all 0, we can also get a differential module with Frobenius structure over $K[\![t]\!]_0$ for the derivation $t\frac{d}{dt}$.)

For example, for the Legendre family of elliptic curves $y^2 = x(x-1)(x-\lambda)$, we take $S = \{0, 1, \infty\}$ and obtain a module corresponding to the hypergeometric equation discussed in the introduction. For $p \neq 2$, that equation admits a Frobenius structure by the above theorem. (For $p = 2$, we cannot make the reduction modulo p generically smooth.)

6 Relationship with zeta functions

The Frobenius structure on a Picard-Fuchs equation can be used to compute zeta functions; this is closely related to the example of Dwork in the introduction. (The condition on λ allows for a unique choice in each residue disc; see exercises.)

Theorem 7. *Retain notation as in Theorem 6, and assume now that $\kappa_K = \mathbb{F}_q$ with $q = p^a$, and that ϕ is a q-power Frobenius lift on $\mathbb{P}^1_{\mathfrak{o}_K}$. Suppose that $\lambda \in \mathfrak{o}_K$ satisfies $\phi(t-\lambda) \equiv 0 \pmod{t-\lambda}$, and suppose that f extends smoothly over the residue disc containing λ. Then*

$$\zeta(f^{-1}(\overline{\lambda}), T) = \prod_{i=0}^{2\dim(f)} \det(1 - T\Phi, (V_i)_\lambda)^{(-1)^{i+1}}.$$

This suggests an interesting strategy for computing zeta functions, advanced by Alan Lauder. Suppose you have in hand the differential module, plus the matrix of the action of Φ on some individual $(V_i)_\lambda$. If you view the equation

$$NA + \frac{dA}{dt} = \frac{d\phi(t)}{dt} A\phi(N)$$

as a differential equation with initial condition provided by $(V_i)_\lambda$, you can then solve for A, and then evaluate at another λ.

More explicitly, let's say for simplicity that $\lambda = 0$ is the starting value. In the open unit disc around 0, you can compute U such that

$$U^{-1}NU + U^{-1}\frac{dU}{dt} = 0$$

and then write down

$$A = UA_0\phi(U^{-1}).$$

This only gives you a power series representation around 0 with radius of convergence 1, which does not give you any way to specialize to, say, $\lambda = 1$.

However, Theorem 6 implies that the entries of A can be written as uniform limits of rational functions with limited denominators. Once you recover a sufficiently good rational function approximation to A, you can specialize at $\lambda = 1$. (I plan to put a more detailed discussion of this technique in the compiled notes.)

7 Notes

Dwork's trick (Corollary 3) still holds without the differential structure, in the following sense. If M is a finite free difference module over $K\{\{t\}\}$ for a Frobenius lift, then there exists an isomorphism $M \cong (M/tM) \otimes_K K\{\{t\}\}$ of difference modules. However, this is somewhat more subtle to prove; see [Ked05c].

The differential operator on a Picard-Fuchs module is also called a *Gauss-Manin connection.*

8 Exercises

1. Prove that for any Frobenius lift ϕ on $K[\![t]\!]_0$, there exists a unique $\lambda \in \mathfrak{m}_K$ such that $\phi(t-\lambda) \equiv 0 \pmod{t-\lambda}$.