p-adic differential equations
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Frobenius antecedents and Frobenius descendants

In this unit, we introduce Dwork's technique of descent along Frobenius (or more exactly, descent along the p-th power map on an affine or projective line) to analyze the generic radius of convergence and subsidiary radii of a differential module.

We retain notation as in the previous unit. In particular, K is a complete nonarchimedean field, and F_ρ is the completion of $K(t)$ for the ρ-Gauss norm for some $\rho > 0$.

1 Why Frobenius?

It may be helpful to review the current state of affairs, to clarify why we need to descend along Frobenius.

Let V be a finite differential module over F_ρ. Then the allowable values of the truncated spectral norm $|D|_{\mathrm{tsp},V}$ are the real numbers greater than or equal to $|d|_{\mathrm{sp},F_\rho} = p^{1/(p-1)}\rho^{-1}$, corresponding to generic radii of convergence less than or equal to ρ.

However, if we want to calculate the truncated spectral norm using the Newton polygon of a twisted polynomial, we cannot distinguish among values less than or equal to the operator norm $|d|_{F_\rho} = \rho^{-1}$. In particular, we cannot use this technique to prove a decomposition theorem for differential modules that separates components of spectral norm between $p^{1/(p-1)}\rho^{-1}$ and ρ^{-1}.

One way one might want to get around this is to consider not d but a high power of d, particularly a p^n-th power. The trouble with this is that iterating a derivation does not give another derivation, but something much more complicated.

Instead, we will try to differentiate with respect to t^{p^n} instead of with respect to t. This will have the effect of increasing the spectral norm, so that we can push it into the range where Newton polygons become useful.

2 p-th roots

But first, we must make some calculations in answer to the following question: if two p-adic numbers are close together, how close are their p-th powers, or their p-th roots?

We observed in the previous unit that when m is a positive integer coprime to p,

$$|t-\eta| < \lambda|\eta| \Leftrightarrow |t^m - \eta^m| < \lambda|\eta|^m \qquad (\lambda \in (0,1)).$$

This breaks down for $m = p$, because a primitive p-th root of unity ζ_p satisfies $|1-\zeta_p| < 1$. The quantities $1-\zeta_p^m$ for $m = 1, \dots, p-1$ are Galois conjugates, so

$$|1-\zeta_p| = \left|\prod_{m=1}^{p-1}(1-\zeta_p^m)\right|^{1/(p-1)} = |p|^{1/(p-1)} = p^{-1/(p-1)}$$

since the product is the derivative of $T^p - 1$ evaluated at $T = 1$.

Lemma 1. *Pick $t, \eta \in K$.*

(a) For $\lambda \in (0,1)$, if $|t-\eta| \leq \lambda|\eta|$, then

$$|t^p - \eta^p| \leq \max\{\lambda^p, p^{-1}\lambda\}|\eta^p| = \begin{cases} \lambda^p|\eta^p| & \lambda \geq p^{-1/(p-1)} \\ p^{-1}\lambda|\eta^p| & \lambda \leq p^{-1/(p-1)}. \end{cases}$$

(b) Suppose $\zeta_p \in K$. If $|t^p - \eta^p| \leq \lambda|\eta^p|$, then there exists $m \in \{0, \dots, p-1\}$ such that

$$|t - \zeta_p^m \eta| \leq \min\{\lambda^{1/p}, p\lambda\}|\eta| = \begin{cases} \lambda^{1/p}|\eta| & \lambda \geq p^{-p/(p-1)} \\ p\lambda|\eta| & \lambda \leq p^{-p/(p-1)}. \end{cases}$$

Moreover, if $\lambda \geq p^{-p/(p-1)}$, we may always take $m = 0$.

We will use repeatedly, and without comment, the fact that

$$\lambda \mapsto \max\{\lambda^p, p^{-1}\lambda\}, \qquad \lambda \mapsto \min\{\lambda^{1/p}, p\lambda\}$$

are strictly increasing functions from $[0,1]$ to itself that are inverse to each other.

Proof. There is no harm in assuming $\zeta_p \in K$ for both parts. For (a), factor $t^p - \eta^p$ as $t - \eta$ times $t - \eta\zeta_p^m$ for $m = 1, \dots, p-1$, and write

$$t - \eta\zeta_p^m = (t - \eta) + \eta(1 - \zeta_p^m).$$

If $|t - \eta| \geq p^{-1/(p-1)}|\eta|$, then $t - \eta$ is the dominant term, otherwise $\eta(1 - \zeta_p^m)$ dominates. This gives the claimed bounds.

For (b), consider the Newton polygon of

$$t^p - \eta^p - c = \sum_{i=0}^{p-1} \binom{p}{i} \eta^i (t-\eta)^{p-i} - c$$

viewed as a polynomial in $t - \eta$. Suppose $|c| = \lambda|\eta^p|$. If $\lambda \geq p^{-p/(p-1)}$, then the terms $(t-\eta)^p$ and c dominate, and all roots have norm $\lambda^{1/p}|\eta|$. Otherwise, the terms $(t-\eta)^p$, $p(t-\eta)\eta^{p-1}$, and c dominate, so one root has norm $p\lambda|\eta|$ and the others are larger; repeating with η replaced by $\zeta_p^m\eta$ for $m = 0, \dots, p-1$ gives p distinct roots, which accounts for all of them. ∎

Corollary 2. *Let $T : K[\![t^p - \eta^p]\!] \to K[\![t - \eta]\!]$ be the substitution $t^p - \eta^p \mapsto ((t-\eta)+\eta)^p - \eta^p$.*

(a) If $f \in K\langle (t^p - \eta^p)/(\lambda|\eta^p|)\rangle$ for some $\lambda \in (0,1)$, then $T(f) \in K\langle (t-\eta)/(\lambda'|\eta|)\rangle$ for $\lambda' = \min\{\lambda^{1/p}, p\lambda\}$.

(b) If $T(f) \in K\langle (t-\eta)/(\lambda|\eta|)\rangle$ for some $\lambda \in (p^{-1/(p-1)}, 1)$, then $f \in K\langle (t^p - \eta^p)/(\lambda'|\eta^p|)\rangle$ for $\lambda' = \lambda^p$.

(c) Suppose K contains a primitive p-th root of unity ζ_p. For $m = 0, \dots, p-1$, let $T_m : K[\![t^p - \eta^p]\!] \to K[\![t - \zeta_p^m\eta]\!]$ be the substitution $t^p - \eta^p \mapsto ((t - \zeta_p^m\eta) + \zeta_p^m\eta)^p - \eta^p$. If for some $\lambda \in (0, p^{-1/(p-1)}]$ one has $T_m(f) \in K\langle (t - \zeta_p^m\eta)/(\lambda|\eta|)\rangle$ for $m = 0, \dots, p-1$, then $f \in K\langle (t^p - \eta^p)/(\lambda'|\eta^p|)\rangle$ for $\lambda' = p^{-1}\lambda$.

3 Moving along Frobenius

Let F'_ρ be the completion of $K(t^p)$ for the ρ^p-Gauss norm, viewed as a subfield of F_ρ, and equipped with the derivation $d' = \frac{d}{dt^p}$. We then have

$$d = \frac{dt^p}{dt} d' = pt^{p-1} d'.$$

Given a finite differential module (V', D') over F'_ρ, we may view $\varphi^* V' = V' \otimes F_\rho$ as a differential module over F_ρ for the derivation $D = pt^{p-1} D' \otimes d$ as a differential

$$D(v \otimes f) = pt^{p-1} D'(v) \otimes f + v \otimes d(f).$$

Lemma 3. *Let (V', D') be a finite differential module over F'_ρ. Then*

$$IR(\varphi^* V') \geq \min\{IR(V')^{1/p}, pIR(V')\}.$$

Proof. For any $\lambda < IR(\varphi^* V')$, any complete extension L of K, and any generic point $t_\rho \in L$ relative to K of norm ρ, $(\varphi^* V') \otimes L\langle (t^p - t_\rho^p)/(\lambda \rho^p) \rangle$ admits a basis of horizontal sections. By Corollary 2(a), $V' \otimes L\langle (t - t_\rho)/(\min\{\lambda^{1/p}, p\lambda\}\rho) \rangle$ does likewise. $\square$

For V a differential module over F_ρ, define the *Frobenius descendant* of V as the module $\varphi_* V$ obtained from V by restriction along $F'_\rho \to F_\rho$, viewed as a differential module over F'_ρ with differential $D' = p^{-1} t^{-p+1} D$. Note that this operation commutes with duals.

For $m = 0, \dots, p-1$, let W_m be the differential module over F'_ρ with one generator v, such that

$$D(v) = \frac{m}{p} t^{-p} v.$$

From the Newton polynomial associated to v, we read off $IR(W_m) = p^{-p/(p-1)}$ for $m \neq 0$. (You may think of the generator v as a proxy for t^m.)

Lemma 4. *(a) For V a differential module over F_ρ, there are canonical isomorphisms $\iota_m : (\varphi_* V) \otimes W_m \cong \varphi_* V$ for $m = 0, \dots, p-1$.*

(b) For V a differential module over F_ρ, a submodule U of $\varphi_* V$ is itself the Frobenius descendant of a submodule of V if and only if $\iota_m(U \otimes W_m) = U$ for $m = 0, \dots, p-1$.

(c) For V' a differential module over F'_ρ, there is a canonical isomorphism

$$\varphi_* \varphi^* V' \cong \bigoplus_{m=0}^{p-1} (V' \otimes W_m).$$

(d) For V a differential module over F_ρ, there is a canonical isomorphism $\varphi^* \varphi_* V \cong V^{\oplus p}$.

(e) For V a differential module over F_ρ, there are canonical bijections $H^i(V) \cong H^i(\varphi_* V)$ for $i = 0, 1$.

(f) For V_1, V_2 differential modules over F_ρ, there is a canonical isomorphism

$$\varphi_* V_1 \otimes \varphi_* V_2 \cong \bigoplus_{m=0}^{p-1} W_m \otimes \varphi_*(V_1 \otimes V_2).$$

Proof. Exercise. □

4 Frobenius antecedents and descendants

Unlike Frobenius descendants, Frobenius antecedents can only be constructed in some cases, namely when the intrinsic radius is sufficiently large.

Theorem 5 (after Christol-Dwork)**.** *Let (V, D) be a finite differential module over F_ρ such that $IR(V) > p^{-1/(p-1)}$. Then there exists a unique differential module (V', D') over F'_ρ such that $V \cong \varphi^* V'$ and $IR(V') > p^{-p/(p-1)}$. For this V', one has in fact $IR(V') = IR(V)^p$.*

The module V' in the theorem is called the *Frobenius antecedent* of V.

Proof of Theorem 5. We may assume $\zeta_p \in K$, as otherwise we may check everything by adjoining ζ_p and then performing a Galois descent at the end.

We first check existence. Since $|D|_{\mathrm{tsp},V} < \rho^{-1}$, for any $x \in V$, we may define an action of $\mathbb{Z}/p\mathbb{Z}$ on V using Taylor series:

$$\zeta_p^m(x) = \sum_{i=0}^{\infty} \frac{(\zeta_p^m t - t)^i}{i!} D^i(x).$$

Take V' to be the fixed space for this action; then V' is an F'_ρ-subspace of V, and the map $\phi^* V' \to V$ is an isomorphism by Hilbert's Theorem 90. (You can also show this explicitly by writing down projectors onto the eigenspaces of V for the $\mathbb{Z}/p\mathbb{Z}$-action.) By applying the $\mathbb{Z}/p\mathbb{Z}$-action to a basis of horizontal sections of V in a generic disc $|t - t_\rho| \leq \lambda\rho$, and invoking Corollary 2(b), we may construct horizontal sections of V' in a generic disc $|t^p - t_\rho^p| \leq \lambda^p \rho^p$. Hence $IR(V') \geq IR(V)^p > p^{-p/(p-1)}$.

To check uniqueness, suppose $V \cong \varphi^* V' \cong \varphi^* V''$ with $IR(V'), IR(V'') > p^{-p/(p-1)}$. By Lemma 4, we have

$$\varphi_* V \cong \oplus_{m=0}^{p-1} (V' \otimes W_m) \cong \oplus_{m=0}^{p-1} (V'' \otimes W_m).$$

For $m = 1, \dots, p-1$, we have $IR(W_m) = p^{-p/(p-1)}$; since $IR(V') > IR(W_m)$, we have $IR(V' \otimes W_m) = p^{-p/(p-1)}$. Since $IR(V'') > p^{-p/(p-1)}$, the factor $V'' \otimes W_0$ must be contained in $V' \otimes W_0$ and vice versa.

For the last assertion, note that the proof of existence gives $IR(V') \geq IR(V)^p$, whereas Lemma 3 gives the reverse inequality. □

Corollary 6. *Let V' be a differential module over F'_ρ such that $IR(V') > p^{-p/(p-1)}$. Then V' is the Frobenius antecedent of $\varphi^* V'$, so $IR(V') = IR(\varphi^* V')^p$.*

The construction of Frobenius antecedents carries over to discs and annuli as follows.

Theorem 7. *Let M be a finite differential module over $K\langle \alpha/t, t/\beta\rangle$ (we may allow $\alpha = 0$), such that $IR(M \otimes F_\rho) > p^{-1/(p-1)}$ for $\rho \in [\alpha, \beta]$ (or equivalently, for $\rho = \alpha$ and $\rho = \beta$). Then there exists a unique differential module M' over $K\langle \alpha^p/t^p, t^p/\beta^p\rangle$ such that $M = M' \otimes K\langle \alpha/t, t/\beta\rangle$ and $IR(M' \otimes F'_\rho) > p^{-p/(p-1)}$ for $\rho \in [\alpha, \beta]$; this M' also satisfies $IR(M' \otimes F'_\rho) = IR(M \otimes F_\rho)^p$ for $\rho \in [\alpha, \beta]$.*

Proof. For existence and the last assertion, use the $\mathbb{Z}/p\mathbb{Z}$-action as in the proof of Theorem 5. (Note that the proof does not apply directly when $\alpha = 0$; we must make a separate calculation on a disc around the origin on which M is trivial.) For uniqueness, apply Theorem 5 for any single $\rho \in [\alpha, \beta]$. □

In the other direction, we can control the intrinsic radius of a Frobenius descendant.

Proposition 8. *Let V be a differential module over F_ρ. Then*

$$IR(\varphi_* V) = \min\{p^{-1} IR(V), p^{-p/(p-1)}\}.$$

Proof. First suppose $IR(V) > p^{-1/(p-1)}$. By Theorem 5, we can write $V \cong \varphi^* V'$ for V' the Frobenius antecedent. By Lemma 4, $\varphi_* V \cong \oplus_{m=0}^{p-1} (V' \otimes W_m)$. In this direct sum, $IR(V' \otimes W_0) = IR(V') > p^{-p/(p-1)}$ and $IR(V' \otimes W_m) = IR(W_m) = p^{-p/(p-1)}$ for $m \neq 0$. Hence $IR(\varphi_* V) = p^{-p/(p-1)}$.

Next suppose $IR(V) \leq p^{-1/(p-1)}$. By Lemma 4, $\varphi^* \varphi_* V \cong V^{\oplus p}$, so by Lemma 3, $IR(V) \geq \min\{IR(\varphi_* V)^{1/p}, p IR(\varphi_* V)\}$. This forces $IR(\varphi_* V) \leq p^{-1} IR(V)$.

In the other direction, for t_ρ a generic point of radius ρ and $\lambda \in (0, p^{-1/(p-1)})$, the module $\varphi_* V \otimes L\langle (t^p - t_\rho^p)/(p^{-1}\lambda\rho^p)\rangle$ splits as the direct sum of $V \otimes L\langle (t - \zeta_p^m t_\rho)/(\lambda\rho)\rangle$ over $m = 0, \dots, p-1$. If $\lambda < IR(V)$, by applying Corollary 2(c), we obtain $IR(\varphi_* V) \geq p^{-1}\lambda$. □

You might be tempted to think that one can run the last part of the previous proof also in the case $IR(V) > p^{-1/(p-1)}$ to prove that $IR(\varphi_* V) \geq IR(V)^p$, which would contradict the first part of the proof. What breaks down in the argument is that in this case, pushing forward a basis of local horizontal sections of V only gives you $(\dim V)$ local horizontal sections of $\varphi_* V$; what they span is precisely the Frobenius antecedent of V.

5 Subsidiary radii and Frobenius

We now refine Proposition 8 to cover subsidiary radii. This will be tremendously important when we study variation of the subsidiary radii in the next unit.

Theorem 9. *Let V be a finite differential module over F_ρ with intrinsic subsidiary radii $s_1, \dots, s_n$. Then the intrinsic subsidiary radii of $\varphi_* V$ comprise the multiset*

$$\bigcup_{i=1}^n \begin{cases} \{s_i^p,\ p^{-p/(p-1)}\ (p-1 \text{ times})\} & s_i > p^{-1/(p-1)} \\ \{p^{-1} s_i\ (p \text{ times})\} & s_i \leq p^{-1/(p-1)}. \end{cases}$$

Proof. It suffices to consider V irreducible. First suppose $IR(V) > p^{-1/(p-1)}$. Let V' be the Frobenius antecedent of V (as per Theorem 5); note that V' is also irreducible. By Lemma 4, $\varphi_* V \cong \oplus_{m=0}^{p-1}(V' \otimes W_m)$. Since each W_m has rank 1, $V' \otimes W_m$ is also irreducible. Since $IR(V') = IR(V)^p$ and $IR(V' \otimes W_m) = p^{-p/(p-1)}$ for $m \neq 0$, we have the claim.

Next suppose $IR(V) \leq p^{-1/(p-1)}$. By Proposition 8, we have $IR(\varphi_* V) = p^{-1} IR(V)$. Let W' be any irreducible subquotient of $\varphi_* V$; then $IR(W') \geq IR(\varphi_* V)$, so Lemma 3 gives

$$IR(\varphi^* W') \geq \min\{IR(W')^{1/p}, pIR(W')\} \geq \min\{IR(\varphi_* V)^{1/p}, pIR(\varphi_* V)\} = IR(V). \quad (1)$$

On the other hand, $\varphi^* W'$ is a subquotient of $\varphi^* \varphi_* V$, which by Lemma 4 is isomorphic to $V^{\oplus p}$. Since V is irreducible, each Jordan-Hölder constituent of $\varphi^* W'$ must be isomorphic to V, yielding $IR(\varphi^* W') = IR(V)$. That forces each inequality in (1) to be an equality; in particular, $IR(W')$ and $IR(\varphi_* V)$ have the same image under the injective map $s \mapsto \min\{s^{1/p}, ps\}$. We conclude that $IR(W') = IR(\varphi_* V) = p^{-1} IR(V)$, proving the claim. □

Corollary 10. *Let $s_1 \leq \cdots \leq s_n$ be the intrinsic subsidiary radii of V.*

(a) For i such that $s_i < p^{-1/(p-1)}$, the product of the pi smallest intrinsic subsidiary radii of $\varphi_* V$ is equal to $p^{-pi} s_1^p \cdots s_i^p$.

(b) For i such that either $i = n$ or $s_{i+1} \geq p^{-1/(p-1)}$, the product of the $pi + (p-1)(n-i)$ smallest intrinsic subsidiary radii of $\varphi_* V$ is equal to $p^{-ni} s_1^p \cdots s_i^p$.

In particular, the product of the intrinsic subsidiary radii of $\varphi_* V$ is $p^{-np} s_1^p \cdots s_n^p$.

Note that both conditions apply when $s_i = p^{-1/(p-1)}$; this will be important later.

6 Decomposition by spectral norm

We now extend the decomposition by spectral norm across the barrier $|d|_{F_\rho}$. This cannot be done using Frobenius antecedents alone: they give no information in case $IR(V) = p^{-1/(p-1)}$.

Proposition 11. *Let V_1, V_2 be irreducible finite differential modules over F_ρ with $IR(V_1) \neq IR(V_2)$. Then $H^1(V_1 \otimes V_2) = 0$.*

Proof. By dualizing if necessary, we can ensure that $IR(V_2) > IR(V_1)$. If $IR(V_1) < p^{-1/(p-1)}$, then any short exact sequence $0 \to V_2 \to V \to V_1^\vee \to 0$ splits by the original decomposition theorem.

Suppose that $IR(V_1) = p^{-1/(p-1)}$. Let V_2' be the Frobenius antecedent of V_2; it is also irreducible, and $IR(V_2') = IR(V_2)^p > p^{-p/(p-1)}$. By Theorem 9, each irreducible subquotient W of $\varphi_* V_1$ satisfies $IR(W) = p^{-p/(p-1)}$; hence $H^1(W \otimes V_2') = 0$ by the previous case, so $H^1(\varphi_* V_1 \otimes V_2') = 0$ by the snake lemma.

By Lemma 4,

$$\begin{aligned} \varphi_* V_1 \otimes \varphi_* V_2 &\cong \oplus_{m=0}^{p-1}(\varphi_* V_1 \otimes W_m \otimes V_2') \\ &\cong (\varphi_* V_1 \otimes V_2')^{\oplus p}. \end{aligned}$$

(The last isomorphism uses the fact that $\varphi_* V_1 \cong \varphi_* V_1 \otimes W_m$.) This yields $H^1(\varphi_* V_1 \otimes \varphi_* V_2) = 0$; since $\varphi_*(V_1 \otimes V_2)$ is a direct summand of $\varphi_* V_1 \otimes \varphi_* V_2$ (again by Lemma 4), $H^1(\varphi_*(V_1 \otimes V_2)) = 0$. By Lemma 4 once more, $H^1(V_1 \otimes V_2) = H^1(\varphi_*(V_1 \otimes V_2)) = 0$.

In the general case, $1 \geq IR(V_2) > IR(V_1)$. If $IR(V_1) > p^{-1/(p-1)}$, then Theorem 5 implies that V_1, V_2 have Frobenius antecedents V_1', V_2', and that any extension $0 \to V_1 \to V \to V_2^\vee \to 0$ itself is the pullback of an extension $0 \to V_1' \to V' \to (V_2')^\vee \to 0$. To show that any extension of the first type splits, it suffices to do so for the second type; that is, we may reduce from V_1, V_2 to V_1', V_2'. By repeating this enough times, we get to a situation where $IR(V_1) \leq p^{-1/(p-1)}$. We may then apply the previous cases. □

From here, the proof of the following theorem is purely formal.

Theorem 12 (Strong decomposition theorem)**.** *Let V be a finite differential module over F_ρ. Then there exists a decomposition*

$$V = \bigoplus_{s \in (0,1]} V_s$$

where every subquotient W_s of V_s satisfies $IR(W_s) = s$.

Proof. We induct on $\dim V$; we need only consider V not irreducible. Choose a short exact sequence $0 \to U_1 \to V \to U_2 \to 0$ with U_2 irreducible. Split $U_1 = \oplus_{s \in (0,1]} U_{1,s}$ where every subquotient W_s of $U_{1,s}$ satisfies $IR(W_s) = s$. For each $s \neq IR(U_2)$, we have $H^1(U_2^\vee \otimes U_{1,s}) = 0$ by repeated application of Proposition 11 plus the snake lemma. Consequently, we have

$$V = V' \oplus \bigoplus_{s \neq IR(U_2)} U_{1,s},$$

where $0 \to U_{1,IR(U_2)} \to V' \to U_2 \to 0$ is exact. □

As with the original decomposition theorem, we obtain the following corollaries.

Corollary 13. *Let V be a finite differential module over F_ρ whose intrinsic subsidiary radii are all less than 1. Then $H^0(V) = H^1(V) = 0$.*

Corollary 14. *With $V = \oplus_{s \in (0,1]} V_s$ as in Theorem 12, we have $H^i(V) = H^i(V_1)$ for $i = 0, 1$.*

This suggests that the difficulties in computing H^0 and H^1 arise in the case of intrinsic generic radius 1. We will pursue a closer study of this case in a later unit.

Corollary 15. *If V_1, V_2 are irreducible and $IR(V_1) < IR(V_2)$, then every irreducible subquotient W of $V_1 \otimes V_2$ satisfies $IR(W) = IR(V_1)$.*

Proof. Decompose $V_1 \otimes V_2 = \oplus_{s \in (0,1]} V_s$ according to Theorem 12; we have $V_s = 0$ whenever $s < IR(V_1)$. If some V_s with $s > IR(V_1)$ were nonzero, then $V_1 \otimes V_2$ would have an irreducible submodule of intrinsic radius greater than $IR(V_1)$, in violation of a result from a previous unit. □

7 Integrality, or lack thereof

It may be useful to keep in mind the following limited integrality result for the intrinsic radius.

Theorem 16. *Let V be a finite differential module over F_ρ with intrinsic subsidiary radii $s_1, \dots, s_n$. Let m be the largest integer such that $s_m = IR(V)$. Then for any nonnegative integer h,*

$$s_1 > p^{p^{-h}/(p-1)} \qquad \Longrightarrow \qquad s_1^m \in |F^\times|^{p^{-h}} \rho^{\mathbb{Z}}.$$

Proof. For $m = 0$, we read this off from a Newton polygon. We reduce from m to $m-1$ by applying φ_* and invoking Theorem 9. □

The exponent p^{-h} cannot be removed; we will give an example to illustrate this in the next unit.

8 Off-centered Frobenius descendants

Since pushing forward along Frobenius does not work well on a disc, we must also consider "off-centered" Frobenius descendants, as follows.

For $\rho \in (p^{-1/(p-1)}, 1]$, let F''_ρ be the completion of $K((t-1)^p - 1)$ under the ρ^p-Gauss norm, or equivalently, under the restriction of the ρ-Gauss norm on $K(t)$. (One could allow $K((t-\mu)^p - \mu^p)$ for any $\mu \in K$ of norm 1, but there is no loss of generality in rescaling t to reduce to the case $\mu = 1$.) For brevity, write $u = (t-1)^p - 1$. Equip F''_ρ with the derivation

$$d'' = \frac{d}{du} = \frac{1}{du/dt} d.$$

Given a differential module V'' over F''_ρ, we may view $\psi^* V'' = V'' \otimes F_\rho$ as a differential module over F_ρ. Given a differential module V over F_ρ, we may view the restriction $\psi_* V$ of V along $F''_\rho \to F_\rho$ as a differential module over F''_ρ.

We may apply Lemma 1 with η replaced by $\eta + 1$, keeping in mind that $|\eta + 1| = 1$ for $|\eta| \leq 1$. This has the net effect that everything that holds for φ also holds for ψ, except that intrinsic radius must be replaced by generic radius.

Theorem 17. *Let (V, D) be a finite differential module over F_ρ such that $R(V) > p^{-1/(p-1)}$. Then there exists a unique differential module (V'', D'') over F''_ρ such that $V \cong \psi^* V''$ and $R(V'') > p^{-p/(p-1)}$. For this V'', one has in fact $R(V'') = R(V)^p$.*

Theorem 18. *Let V be a finite differential module over F_ρ with extrinsic subsidiary radii $s_1, \dots, s_n$. Then the subsidiary radii of $\psi_* V$ comprise the multiset*

$$\bigcup_{i=1}^n \begin{cases} \{s_i^p,\ p^{-p/(p-1)}\ (p-1 \text{ times})\} & s_i > p^{-1/(p-1)} \\ \{p^{-1} s_i\ (p \text{ times})\} & s_i \leq p^{-1/(p-1)}. \end{cases}$$

Note that one cannot expect Theorem 18 to hold for $\rho < p^{-1/(p-1)}$, as in that case $p^{-p/(p-1)}$ is too big to appear as a subsidiary radius of $\psi_* V$.

9 Notes

Lemma 1 is taken from [Ked05, §5.3] with some typos corrected.

The Frobenius antecedent theorem of Christol-Dwork [CD94, Théorème 5.4] is slightly weaker than the one given here: it only applies for $IR(V) > p^{-1/p}$. The trouble is that they use cyclic vectors, which create some regular singularities which they only eliminate under the stronger hypothesis. Theorem 5 as stated there first appears in [Ked05, Theorem 6.13], except that there uniqueness is only given if $IR(V') \geq IR(V)^p$.

To the best of my knowledge, the study of Frobenius descendants is original to this presentation; in particular, Theorems 9 and 18 are original. The strong decomposition theorem (Theorem 12) is also original; we do not know of a proof without Frobenius descendants.

10 Exercises

1. Prove Lemma 4.

2. Prove that for any finite differential module V' over F'_ρ with $IR(V') > p^{-p/(p-1)}$, $H^0(V') = H^0(\varphi^* V')$.